CONSISTENCY AND FEASIBILITY OF APPROXIMATE DECONVOLUTION MODELS OF TURBULENCE

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Abstract. We consider the time averaged consistency error in approximate deconvolution LES models. One result is that the time averaged consistency error τ_0 of the zeroth order model converges to zero uniformly in the kinetic viscosity and in the Reynolds number. We next give consistency error bounds for the higher oder models, showing their consistency errors $\tau_N \to 0$ rapidly for the averaging radius well within the inertial range. The error in the higher order models also decreases as the length scale L increases.

Key words. large eddy simulation, approximate deconvolution model, turbulence

AMS subject classiøcation.

1. Introduction. Direct numerical simulation of turbulent Æows of incompressible, viscous Æuids is often not computationally economical or even feasible. Thus, various turbulence models are used for simulations seeking to predict Æow statistics or averages. In LES (large eddy simulation) the evolution of local, spatial averages is sought. Broadly, there are two types of LES models of turbulence: descriptive or phenomenological models (e.g., eddy viscosity models) and predictive models (considered herein). The accuracy of a model (meaning $||averagedNSEsolution - LESsolution||$) can be assessed in several experimental and analytical ways. One important approach (for which there are currently few results) is to study analytically the model's consistency error (deøned precisely below) as a function of the averaging radius δ and the Reynolds number Re. The inherent diŒculties are that (i) consistency error bounds for inønitely smooth functions hardy address essential features of turbulent Æows such as irregularity and richness of scales, and (ii) worst case bounds for general weak solutions of the Navier Stokes equations are so pessimistic as to yield little insight. However, it is known that after time or ensemble averaging, turbulent velocity øelds are often observed to have intermediate regularity as predicted by the Kolmogorov theory (often called the K41 theory), see, for example, [F95],[BIL04],[P00], [S01]. This case is often referred to as homogeneous isotropic turbulence and various norms of Æow quantities can be estimated in this case using the K41 theory, Plancherel's Theorem and spectral integration. We mentioned Lilly's famous paper [L67] as an early and important example.

In this report we consider this third way begun in [LL04b]: consistency error bounds are developed for time averaged, fully developed, homogeneous, isotropic turbulence. Such bounds are inherently interesting and they also answer two important related questions of accuracy and feasibility of LES. How small must δ be with respect to Re to have the average consistency error $<< O(1)$? Can consistency error $<< O(1)$ be attained for the cutoœ length-scale δ within the inertial range?

Let the velocity $u(x,t) = u_j(x_1, x_2, x_3, t)$, $(j = 1, 2, 3)$ and pressure $p(x,t) = p(x_1, x_2, x_3, t)$ be a weak solution to the underlying Navier Stokes equations (NSE for

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short)

$$u_t + u \cdot \nabla u - \nu \triangle u + \nabla p = f, \ and \ \nabla \cdot u = 0, \ in \ \Omega \times (0,T), \tag{1.1}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, f is the body force, p is the pressure, $\Omega = (0,L)^3$ is the Æow domain. The above Navier-Stokes equations are supplemented by the initial condition, the usual pressure normalization condition

$$u(x,0) = u_0(x), \ and \ \int_\Omega p dx = 0, \tag{1.2}$$

and appropriate boundary conditions; our estimates are for the case of periodic boundary conditions with zero mean imposed upon the velocity and all data

$$u(x_j + L, t) = u(x_j, t), and \int_\Omega u(x,t) dx = 0, \tag{1.3}$$

$$where \int_\Omega u_0(x) dx = 0, \ and \ \int_\Omega f(x,t) dx = 0, \ for \ 0 \le t \le T. \tag{1.4}$$

We study a model for spacial averages of the Æuid velocity with the following differential ølter. Let δ denote the averaging radius; given ϕ, its average, denoted $\overline{\phi}$, is the solution of the following boundary value problem under L-periodic boundary conditions:

$$A\overline{\phi} := -(\frac{\delta}{L})^2 \triangle \overline{\phi} + \overline{\phi} = \phi. \tag{1.5}$$

Averaging the NSE shows that the true Æow averages satisfy the (non-closed) equations

$$\overline{u}_t + \nabla \cdot (\overline{u\, u}) - \nu \triangle \overline{u} + \nabla \overline{p} = \overline{f}, \ and \ \nabla \cdot \overline{u} = 0. \tag{1.6}$$

The zeroth order model arises from $u \simeq \overline{u} + O(\delta^2)$, giving $\overline{u\, u} \simeq \overline{\overline{u}\, \overline{u}} + O(\delta^2)$. Calling w, q the resulting approximations to $\overline{u}, \overline{p}$, we obtain the model studied in [LL03],[LL04]:

$$w_t + \nabla \cdot (\overline{w\, w}) - \nu \triangle w + \nabla q = \overline{f}, \ and \ \nabla \cdot w = 0. \tag{1.7}$$

This zeroth order model's consistency error τ_0 is given by:

$$\tau_0 := \overline{u}\, \overline{u} - u\, u. \tag{1.8}$$

The model's error, $\overline{u} - w$, satisøes $e(0,x) = 0, \nabla \cdot e = 0$ and

$$(\overline{u} - w)_t + \nabla \cdot (\overline{u\, u} - \overline{w\, w}) - \nu \triangle (\overline{u} - w) + \nabla (\overline{p} - q) = \nabla \cdot \overline{\tau_0} \tag{1.9}$$

which is driven only by the model's consistency error τ_0. Since the model is stable and stable to perturbations, [LL04] , the accuracy of the model is governed by the size of various norms of its consistency error tensor τ_0.

The above example is the simplest (hence zeroth order) model in many families of LES models. We consider herein a family of Approximate Deconvolution Models (or ADM's) whose use in LES was pioneered by Stolz and Adams in a series of papers,[AS01], [AS99]. The size of the Nth models consistency error tensor directly determines the model's accuracy for these higher order model's as well, [DE04].Given the van Cittert, [BB98], approximate deconvolution operator G_N $(N = 0, 1, 2, ...)$ satisfying

$$u = G_N\overline{u} + O(\delta^{2N+2}), \text{ for smooth } u, \tag{1.10}$$

the models studied by Adams and Stolz are given by

$$w_t + \nabla \cdot (\overline{G_N w \; G_N w}) - \nu \triangle w + \nabla q + w' = \overline{f}, \; and \; \nabla \cdot w = 0. \tag{1.11}$$

The w'term is included to damp strongly the temporal growth of the Æuctuating component of w driven by noise, numerical errors, inexact boundary conditions and so on. Herein, we drop the w' term[1], select the averaging operator to be the above diœerential ølter and (following Adams and Stolz) choose G_N to be the van Cittert approximation, [BB98],

$$G_N\phi := (-1)^{N+1}\sum_{n=0}^{N}[I - A^{-1}]^n\phi. \tag{1.12}$$

For example, the induced closure model's corresponding to $N = 0$ and 1 are

$$G_0\overline{u} = \overline{u}, so \; \overline{u\;u} \simeq \overline{\overline{u}\;\overline{u}} + O(\delta^2), \tag{1.13}$$

$$G_1\overline{u} = 2\overline{u} - \overline{\overline{u}}, so \; \overline{u\;u} \simeq \overline{(2\overline{u} - \overline{\overline{u}})\;(2\overline{u} - \overline{\overline{u}})} + O(\delta^4). \tag{1.14}$$

To present the results, let $< \cdot >$ denote time averaging (deøned precisely in section 2), δ the averaging radius used in the LES model, L the global length scale, Re the Reynolds number and U the characteristic velocity used to non-dimensionalize the equations by $U = time\ average\ of\{\ \frac{1}{L^3}||u(x,t)||^2_{L^2(\Omega)}\}^{\frac{1}{2}}$. The normal setting is characteristic velocity $U = O(1)$, $L = O(1)$,and the Reynolds number, Re, large because the kinetic viscosity, ν, is small. There are other cases in which Re is large due to L (such as geophysical Æows, [Lew97]) or U (in wind tunnels for example).

1.1. The Zeroth Order Model. Consider ørst the case of the zeroth order model, (1.7) above. For the case $N = 0$ and for smooth u, it is easy to show that $\tau_0 = O((\frac{\delta}{L})^2)$.Indeed, simple estimates give $||u-\overline{u}||_{L^2(\Omega)} \le (\frac{\delta}{L})^2||\triangle u||_{L^2(\Omega)}$, and thus since $\tau = \overline{u}\ (\overline{u} - u) + (\overline{u} - u)u$, it follows immediately that

$$||\tau_0||_{L^1(\Omega)} \le 2||u||_{L^2(\Omega)}(\frac{\delta}{L})^2||\triangle u||_{L^2(\Omega)}, and < ||\tau_0||_{L^1(\Omega)} >\le C\alpha^{\frac{1}{2}}\frac{U^2}{L^{\frac{1}{2}}}\,\mathrm{Re}^{\frac{5}{4}}(\frac{\delta}{L})^2 \tag{1.15}$$

While relevant in smooth regions of transitional Æows, this smoothness, $\triangle u \in L^2(\Omega)$, needed does not describe the typical case of turbulent Æows. Next we show in Section 3 that

$$< ||\tau_0||_{L^1(\Omega)} \;\; >\le \mathrm{Re}^{\frac{1}{2}}\; L^{\frac{1}{2}}U^4\frac{\delta}{L}. \tag{1.16}$$

These two estimates, (1.15) and (1.16) next, are not suŒciently sharp to draw useful conclusions at higher Reynolds numbers (see Section 4). For example, this estimate

[1] The consistency error induced by adding the w' term is smaller than that of the nonlinear term. While it does aœect the model's dynamics, it does not aœect the overall consistency error estimate.

suggests the zeroth order model is $O(\frac{\delta}{L})$ accurate only for $\frac{\delta}{L} << \mathrm{Re}^{-\frac{1}{2}}$. In our third estimate, using the K-41 phenomenology and spectral integration, we show, remarkably, the time averaged modeling consistency error is $O((\frac{\delta}{L})^{\frac{1}{3}})$ uniformly in the Reynolds number Re and the kinematic viscosity ν:

$$< ||\tau_0||_{L^1(\Omega)} >\simeq 6.8 L^{\frac{7}{6}} U^2 (\frac{\delta}{L})^{\frac{1}{3}}. \tag{1.17}$$

To illustrate the improvement of (1.7) over (1.6), supressing all parameters except $\frac{\delta}{L}$ and Re, (1.6), and (1.7) together imply

$$< ||\tau_0||_{L^1(\Omega)} >\simeq C \min\{(\frac{\delta}{L})^{\frac{1}{3}}, \mathrm{Re}^{\frac{1}{2}}(\frac{\delta}{L})\}. \tag{1.18}$$

The crossover point when (1.6) becomes sharper than (1.7) in (1.8) is when $(\frac{\delta}{L})^{\frac{1}{3}} \simeq \mathrm{Re}^{\frac{1}{2}}(\frac{\delta}{L})$, or equivalently $(\frac{\delta}{L}) \simeq \mathrm{Re}^{-\frac{3}{4}}$, i.e., only when the Æow is fully resolved!

1.2. The General Approximate Deconvolution Model. In Section 3 the above techniques are adapted to the general case. The pointwise error in deconvolution by G_N was calculated explicitly via the Neumann lemma in Lemma 2.3 in Dunca and Epshteyn [DE04],

$$u - G_N \overline{u} = (-1)^{N+1} (\frac{\delta}{L})^{2N+2} \triangle^{N+1} \overline{u}. \tag{1.19}$$

The model's consistency error, τ_N, is given by

$$\tau_N := G_N \overline{u} G_N \overline{u} - u\, u \tag{1.20}$$

Adapting the ideas in the zeroth order case and using this last formula, in section 3 we give a ørst estimate of the model's consistency error τ_N

$$< ||\tau_N||_{L^1(\Omega)} >\leq \frac{2C_1 \alpha^{\frac{1}{2}}}{(2N+\frac{4}{5})^{\frac{1}{2}}} \frac{U^2}{L^{N-\frac{1}{2}}} (\frac{\delta}{L})^{2N+2} \mathrm{Re}^{\frac{3}{4}N+\frac{1}{2}}. \tag{1.21}$$

A second sharper estimate is then proven; this estimate takes the general form:

$$< ||\tau_N||_{L^1(\Omega)} >\leq C(N,\alpha,U,L)[\mathrm{Re}^{\frac{3}{4}N-\frac{1}{4}}(\frac{\delta}{L})^{2N+1} + (\frac{\delta}{L})^{N+\frac{4}{3}}], \tag{1.22}$$

where the precise dependence upon N, α, U, L is given in the derivation.

The impact of this and the previous estimates on practical issues in LES is considered in section 4. We shall see that these two estimates are strictly better than the $N = 0$ case with respect to both dependence on L and the resolution required for an accurate LES.

2. The K-41 formalism. The most important components of the K-41 theory are the time (or ensemble) averaged energy dissipation rate, ε, and the distribution of the Æows kinetic energy across wave numbers, $E(k)$. Let $< \cdot >$ denote long time averaging

$$< \phi > (x) := \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T \phi(x,t) dt. \tag{2.1}$$

Time averaging is the original approach to turbulence of Reynolds, [R95]. It satisøes the following Cauchy-Schwartz inequality

$$< (\phi, \psi)_{L^2(\Omega)} > \quad \leq \quad < ||\phi||^2_{L^2(\Omega)} >^{\frac{1}{2}} < ||\psi||^2_{L^2(\Omega)} >^{\frac{1}{2}} . \tag{2.2}$$

This follows, for example, by applying the usual C-S inequality on $\Omega x(0, T)$ followed by taking limits or from the connection with the inner product on the space of Besicovitch almost periodic functions, e.g., [Z85],[L84], [CB89].

Given the velocity øeld of a particular Æow, $u(x, t)$, the (time averaged) energy dissipation rate of that Æow is deøned to be

$$\varepsilon := \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T \int_\Omega \nu |\nabla u(x, t)|^2 dx\ dt. \tag{2.3}$$

If $\widehat{u}(\mathbf{k}, t)$ denotes the Fourier transform of $u(x, t)$ where $\mathbf{k}$ is the wave-number vector and $k = |\mathbf{k}|$ is its magnitude, then Plancherel's Theorem implies that the kinetic energy in u can be evaluated in physical space or in wave number space using the Fourier transform $\widehat{u}$ of u

$$\frac{1}{2}||u||^2_{L^2(\Omega)} = \frac{1}{2} \int_\Omega |u(x, t)|^2 dx = \frac{1}{2} \int_{\mathbf{R}^3} |\widehat{u}(\mathbf{k},t)|^2 d\mathbf{k}. \tag{2.4}$$

Time averaging and rewriting the last integral in spherical coordinates gives

$$< \frac{1}{2}||u||^2_{L^2(\Omega)} >= \int_0^\infty E(k)dk, \quad where\ E(k) := \int_{|\mathbf{k}|=k} \frac{1}{2} |\widehat{< u >}(\mathbf{k}, t)|^2 d\sigma. \tag{2.5}$$

The case of homogeneous, isotropic turbulence includes the assumption that (after time or ensemble averaging) $\widehat{u}(\mathbf{k})$ depends only on k and thus not the angles θ or φ. Thus, in this case,

$$E(k) = 2\pi k^2 |\widehat{< u >}(k)|^2. \tag{2.6}$$

Further, the K-41 theory states that at high enough Reynolds numbers there is a range of wave numbers

$$0 < k_{\text{min}} := U\nu^{-1} \leq k \leq \varepsilon^{\frac{1}{4}} \nu^{-\frac{3}{4}} =: k_{\text{max}} < \infty, \tag{2.7}$$

known as the inertial range, beyond which the kinetic energy in u is negligible, and in this range

$$E(k) \doteq \alpha \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \tag{2.8}$$

where $\alpha (\simeq 1.4)$ is the universal Kolmogorov constant, k is the wave number and ε is the particular Æow's energy dissipation rate. The energy dissipation rate ε is the only parameter which diæers from one Æow to another. Outside the inertial range we still have $E(k) \leq \alpha \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ since$E(k) \simeq 0$ for $k \geq k_{\text{max}}$ and $E(k) \leq E(k_{\text{min}})$ for $k \leq k_{\text{min}}$. The expression (2.8) for E(k) is the fundamental assumption underlying our consistency error estimates.

3. Estimation of the consistency error. First note in all cases, the consistency error depends upon estimates of $u - G_N\overline{u}$ because

$$\tau_N = G_N\overline{u}\ G_N\overline{u} - u\ u = (G_N\overline{u} - u)G_N\overline{u} + u(G_N\overline{u} - u), N = 0, 1, 2, \cdots. \tag{3.1}$$

Consider τ_N. By the time averaged Cauchy-Schwarz inequality, and stability bounds for G_N we have

$$< ||\tau_N||_{L^1(\Omega)} > \leq (1 + ||G_N||) < ||u||^2_{L^2(\Omega)} >^{\frac{1}{2}} < ||u - G_N\overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}} \tag{3.2}$$

Thus, estimates for the consistency error in $L^1(\Omega)$ Æow from the above estimates of $||u - G_N\overline{u}||_{L^2(\Omega)}$ and later estimates of $< ||u - G_N\overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}}$.

3.1. Estimates for the Zeroth Order Model. By multiplying (1.5) by $\overline{\phi}$, integrating by parts over Ω and using a CBS inequality on the right hand side, it follows readily that the averaging process is stable and smoothing in the sense

$$||\overline{\phi}||,\ 2\frac{\delta}{L}||\nabla\overline{\phi}||,\ \text{and}\ \frac{1}{2}(\frac{\delta}{L})^2||\triangle\overline{\phi}||\ \leq ||\phi||. \tag{3.3}$$

Denote the averaging error by $\Phi = (\phi - \overline{\phi})$.Using the equation $-(\frac{\delta}{L})^2\triangle\Phi + \Phi = -(\frac{\delta}{L})\triangle\phi$, the following error bounds for Φ follow in much the same ways as the above stability bounds

$$||\phi - \overline{\phi}|| \leq \frac{1}{\sqrt{2}}(\frac{\delta}{L})||\nabla\phi||,\ ||\nabla(\phi - \overline{\phi})|| \leq \frac{1}{\sqrt{2}}(\frac{\delta}{L})||\triangle\phi||,\ \textit{and}\ ||\phi - \overline{\phi}|| \leq (\frac{\delta}{L})^2||\triangle\phi||. \tag{3.4}$$

Consider τ. By the time-averaged Cauchy-Schwartz inequality and the above stability bounds we have

$$< ||\tau_0||_{L^1(\Omega)} > \leq 2 < ||u||^2_{L^2(\Omega)} >^{\frac{1}{2}} < ||u - \overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}} \tag{3.5}$$

Estimates for τ thus follow from estimates for $||u - \overline{u}||_{L^2(\Omega)}$ and $< ||u - \overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}}$.

It is possible to get a very quick estimate of $< ||\tau_0||_{L^1(\Omega)} >$ by scaling, dimensional analysis and using known PDE estimates as follows. Holder's inequality and the above simple estimate $||u - \overline{u}||_{L^2(\Omega)} \leq (\frac{\delta}{L})||\nabla u||$ give

$$< ||\tau_0||_{L^1(\Omega)} > \leq 2L^{\frac{3}{2}}U(\frac{\delta}{L})\nu^{-\frac{1}{2}} < \nu||\nabla u||^2_{L^2(\Omega)} >^{\frac{1}{2}} \leq 2\nu^{-\frac{1}{2}}L^{\frac{3}{2}}U(\frac{\delta}{L})\varepsilon^{\frac{1}{2}}. \tag{3.6}$$

It is known for many turbulent Æows that the energy dissipation rate ε scales like $C_1\frac{U^3}{L}$. This estimate follows for homogeneous, isotropic turbulence from the K-41 formalism, [F95], and has been proven as an upper bound directly from the Navier Stokes equations for quite general turbulent shear Æows, [CD92] ,[W97] . Using this upper bound for ε gives the bound (1.16)

$$< ||\tau_0||_{L^1(\Omega)} > \leq C_1L^{\frac{1}{2}}U^{\frac{7}{2}}\,\mathrm{Re}^{\frac{1}{2}}\,\frac{\delta}{L}. \tag{3.7}$$

This bounds is very rough and pessimistic since it requires $\frac{\delta}{L} << \mathrm{Re}^{-\frac{1}{2}}$ for accuracy, see section 4. Remarkably, this estimate is improvable to one uniform in the Reynolds number in the case of homogeneous, isotropic turbulence.

The related (1.15) estimate is obtained by using instead

$$< ||\tau_0||_{L^1(\Omega)} >\leq 2 < ||u||^2_{L^2(\Omega)} >^{\frac{1}{2}} < ||u-\overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}} \leq 2UL^{\frac{3}{2}}(\frac{\delta}{L})^2 < ||\triangle u||^2 >^{\frac{1}{2}} . \tag{3.8}$$

The term $< ||\triangle u||^2 >^{\frac{1}{2}}$ can be estimated in the case of homogeneous, isotropic turbulence using spectral integration as follows

$$< ||\triangle u||^2 >= \int_{k_0}^{k_{\max}} k^4 E(k) dk \leq \alpha\varepsilon^{\frac{2}{3}} \int_0^{k_{\max}} k^{\frac{7}{3}} dk = .3\alpha\varepsilon^{\frac{2}{3}}(\varepsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}})^{\frac{10}{3}}. \tag{3.9}$$

Using the estimate $\varepsilon \leq C\frac{U^3}{L}$ and rearranging the resulting RHS into terms involving the Reynolds number gives

$$< ||\triangle u||^2 >^{\frac{1}{2}} \leq C\alpha^{\frac{1}{2}}\frac{U}{L^2}\,\mathrm{Re}^{\frac{5}{4}}(\frac{\delta}{L})^2, \tag{3.10}$$

which gives

$$< ||\tau_0||_{L^1(\Omega)} > \leq C\alpha^{\frac{1}{2}}\frac{U^2}{L^{\frac{1}{2}}}\,\mathrm{Re}^{\frac{5}{4}}(\frac{\delta}{L})^2. \tag{3.11}$$

This is an asymptotically higher power of $\frac{\delta}{L}$ for moderate Reynolds numbers but it yields the consistency condition $\frac{\delta}{L} << \mathrm{Re}^{-\frac{5}{8}}$ which is worse than the preceding one.

The sharper bound (1.17) is proven as follows. Under the K-41 formalism we can write

$$< ||u-\overline{u}||^2_{L^2(\Omega)} > \leq \int_{k_0}^{k_{\max}} (1 - \frac{1}{(\frac{\delta}{L})^2 k^2 + 1}) E(k) dk, \tag{3.12}$$

where $(0 <)k_0 (\leq k_{min})$ is the smallest frequency, and $k_{max} (= \varepsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}})$ the largest frequency. Over the inertial range $E(k) = \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$ and outside it $E(k) \leq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$. Thus, we can write

$$< ||u-\overline{u}||^2_{L^2(\Omega)} > \leq 2\alpha\varepsilon^{\frac{2}{3}} \int_{k_0}^{k_{\max}} \frac{(\frac{\delta}{L})^2 k^2}{(\frac{\delta}{L})^2 k^2 + 1} k^{-\frac{5}{3}} dk =: I. \tag{3.13}$$

The remainder of the work in (1.17) is direct estimation of the above integral. The integral I requires dioeerent treatments for small and large wave numbers. We shall thus estimate the two cases separately

$$I := I_{low} + I_{high}, \text{where } I_{low} = \int_{k_0}^{\frac{L\pi}{\delta}} ...dk, \text{and } I_{high} = \int_{\frac{L\pi}{\delta}}^{k_{high}} ...dk. \tag{3.14}$$

For the low frequency components we have

$$\frac{(\frac{\delta}{L})^2 k^2}{(\frac{\delta}{L})^2 k^2 + 1} \leq (\frac{\delta}{L})^2 k^2, \text{for } 0 \leq k \leq \frac{L\pi}{\delta}, \tag{3.15}$$

and thus

$$I_{low} \leq 2(\frac{\delta}{L})^2 \alpha\varepsilon^{\frac{2}{3}} \int_{k_0}^{\frac{L\pi}{\delta}} k^{\frac{1}{3}} dk \leq 2(\frac{\delta}{L})^2 \alpha\varepsilon^{\frac{2}{3}} \int_0^{\frac{L\pi}{\delta}} k^{\frac{1}{3}} dk = \frac{3}{2}\pi^{\frac{4}{3}}\alpha\varepsilon^{\frac{2}{3}}(\frac{\delta}{L})^{\frac{2}{3}}. \tag{3.16}$$

Consider the second integral; over the high frequency components

$$I_{high} = 2(\frac{\delta}{L})^2 \alpha \varepsilon^{\frac{2}{3}} \int_{\frac{L\pi}{\delta}}^{k_{\max}} \frac{(\frac{\delta}{L})^2 k^2}{(\frac{\delta}{L})^2 k^2 + 1} k^{-\frac{5}{3}} dk. \tag{3.17}$$

With the obvious change of variables $k' = \frac{\delta}{L}k$, the integral becomes

$$I_{high} = 2(\frac{\delta}{L})^{\frac{2}{3}} \alpha \varepsilon^{\frac{2}{3}} \int_{\pi}^{\frac{\delta}{L}k_{\max}} \frac{(k')^{\frac{1}{3}}}{(k')^2 + 1} dk' \leq 2(\frac{\delta}{L})^{\frac{2}{3}} \alpha \varepsilon^{\frac{2}{3}} \int_{\pi}^{\frac{\delta}{L}k_{\max}} \frac{(k')^{\frac{1}{3}}}{2(k')^2} dk'. \tag{3.18}$$

Replace the above integral with one over $[\pi, \infty)$. This leads to the simpliøed and negligently less sharp upper estimate for I_{high} :

$$I_{high} \leq 3\pi^{-\frac{2}{3}} \alpha \varepsilon^{\frac{2}{3}} (\frac{\delta}{L})^{\frac{2}{3}}. \tag{3.19}$$

Adding these estimates we obtain

$$I \leq 8.3 \alpha \varepsilon^{\frac{2}{3}} (\frac{\delta}{L})^{\frac{2}{3}}. \tag{3.20}$$

Using this bound for I in $< ||\tau_0||_{L^1(\Omega)} > \leq 2UL^{\frac{3}{2}} I^{\frac{1}{2}}$ gives the following estimate for the model's consistency error

$$< ||\tau_0||_{L^1(\Omega)} > \simeq 6.8 U L^{\frac{3}{2}} \varepsilon^{\frac{1}{3}} (\frac{\delta}{L})^{\frac{1}{3}}. \tag{3.21}$$

Using the estimate for $\epsilon \leq C_1 \frac{U^3}{L}$, (independent of Re and ν), we obtain the claimed estimate (1.17)

$$< ||\tau_0||_{L^1(\Omega)} > \leq C_1 6.8 U^2 L^{\frac{7}{6}} (\frac{\delta}{L})^{\frac{1}{3}}. \tag{3.22}$$

This is remarkable in that it predicts the models consistency error to approach zero uniformly in the Reynolds number.

3.2. The General Approximate Deconvolution Model. The analysis in the case $N = 1, 2, 3, \cdots$ follows the zeroth order case using stability of G_N and the estimates

$$< ||\tau_N||_{L^1} > \leq (1 + ||G_N||) < ||u||_{L^2}^2 >^{\frac{1}{2}} < ||u - G_N \overline{u}||_{L^2}^2 >^{\frac{1}{2}}, \tag{3.23}$$

$$u - G_N \overline{u} = (-1)^{N+1} (\frac{\delta}{L})^{2N+2} \triangle^{N+1} \overline{u} \tag{3.24}$$

Indeed, beginning with $< ||u||_{L^2}^2 >^{\frac{1}{2}} \leq UL^{\frac{3}{2}}$, we have

$$< ||\tau_N||_{L^1(\Omega)} > \leq (1 + ||G_N||) UL^{\frac{3}{2}} < ||u - G_N \overline{u}||_{L^2(\Omega)}^2 >^{\frac{1}{2}} . \tag{3.25}$$

First, note that by the spectral mapping theorem

$$||G_N|| = \sum_{n=0}^{N} \lambda_{\max} (I - A^{-1})^n = \sum_{n=0}^{N} (I - \frac{1}{\lambda_{\max}})^n = N + 1. \tag{3.26}$$

As in subsection 3.1, we use spectral integration to evaluate the deconvolution approximation's error as follows

$$I :=< ||u - G_N\overline{u}||^2_{L^2(\Omega)} >= (\frac{\delta}{L})^{4N+4} < ||\triangle^{N+1}\overline{u}||^2_{L^2(\Omega)} >= \tag{3.27}$$

$$(\frac{\delta}{L})^{4N+4} \int_{k_0}^{k_{\max}} \frac{k^{2N+2}}{1+(\frac{\delta}{L})^2k^2} E(k)dk. \tag{3.28}$$

Since $E(k) \leq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$ we have

$$I \leq (\frac{\delta}{L})^{4N+4}\alpha\varepsilon^{\frac{2}{3}} \int_0^{k_{\max}} \frac{k^{2N+2-\frac{5}{3}}}{1+(\frac{\delta}{L})^2k^2}dk. \tag{3.29}$$

Using the fact that the denominator of the integrand is bounded below by 1 gives the simple estimate

$$I \leq (\frac{\delta}{L})^{4N+4}\alpha\varepsilon^{\frac{2}{3}} \frac{1}{2N+\frac{4}{3}}(\varepsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}})^{2N+\frac{4}{3}} \tag{3.30}$$

As $\varepsilon \leq C_1\frac{U^3}{L}$, regrouping the RHS to write it in terms of the Reynolds number gives the ørst bound on I

$$I \leq \frac{C_1\alpha}{2N+\frac{4}{3}}(\frac{\delta}{L})^{4N+4}\frac{U^2}{L^{2N+2}}\,\mathrm{Re}^{\frac{3}{2}N+1}. \tag{3.31}$$

Thus, as $< ||\tau_N||_{L^1(\Omega)} >\leq (N+1)UL^{\frac{3}{2}}I^{\frac{1}{2}}$, we have a simple bound

$$< ||\tau_N||_{L^1(\Omega)} >\leq (N+1)(\frac{C_1\alpha}{2N+\frac{4}{3}})^{\frac{1}{2}}(\frac{\delta}{L})^{2N+2}\frac{U^2}{L^{N-\frac{1}{2}}}\,\mathrm{Re}^{\frac{3}{4}N+\frac{1}{2}}. \tag{3.32}$$

The above estimate of I is improvable since I has diœerent asymptotics for low and high wave-numbers. As in the zeroth order case, split

$$I := I_{low} + I_{high}, \text{where } I_{low} = \int_{k_0}^{\frac{L\pi}{\delta}} ...dk, \text{and } I_{high} = \int_{\frac{L\pi}{\delta}}^{k_{high}} ...dk. \tag{3.33}$$

For the low frequencies we have

$$I_{low} \leq (\frac{\delta}{L})^{4N+4}\alpha\varepsilon^{\frac{2}{3}} \int_0^{\frac{\pi}{\delta}} \frac{k^{2N+2-\frac{5}{3}}}{1+(\frac{\delta}{L})^2k^2}dk \leq (\frac{\delta}{L})^{4N+4}\alpha\varepsilon^{\frac{2}{3}} \int_0^{\frac{\pi}{\delta}} k^{2N+2-\frac{5}{3}}dk. \tag{3.34}$$

Thus,

$$I_{low} \leq (\frac{\delta}{L})^{2N+2+\frac{2}{3}}\alpha\varepsilon^{\frac{2}{3}}\frac{\pi^{2N+\frac{4}{3}}}{2N+\frac{4}{3}}. \tag{3.35}$$

For the high wave numbers we have

$$I_{high} \leq (\frac{\delta}{L})^{4N+4}\alpha\varepsilon^{\frac{2}{3}} \int_{\frac{\pi}{\delta}}^{k_{\max}} \frac{k^{2N+2-\frac{5}{3}}}{(\frac{\delta}{L})^2k^2}dk, \tag{3.36}$$

or,

$$I_{high} \leq (\frac{\delta}{L})^{2N+2+\frac{2}{3}} \alpha \varepsilon^{\frac{2}{3}} \frac{1}{4N-\frac{4}{3}} [((\frac{\delta}{L}) \varepsilon^{\frac{1}{4}} \nu^{-\frac{3}{4}})^{2N-\frac{2}{3}} - \pi^{2N-\frac{2}{3}}]. \tag{3.37}$$

Using the bound $\varepsilon \leq C_1 \frac{U^3}{L}$, collecting the two estimates for $I := I_{low} + I_{high}$ and writing the result in terms of the Reynolds number gives the sharper (and longer) estimate

$$< ||\tau_N||_{L^1(\Omega)} > \leq \frac{N+1}{(4N-\frac{4}{3})^{\frac{1}{2}}} C_1^{\frac{1}{8}} \alpha^{\frac{1}{2}} U^{-\frac{3}{4}N+\frac{13}{8}} L^{-\frac{3}{4}N+\frac{13}{8}} \mathrm{Re}^{\frac{3}{4}N-\frac{1}{4}} (\frac{\delta}{L})^{2N+1} + \tag{3.38}$$

$$+(N+1)(\frac{\pi^{2N+\frac{4}{3}}}{2N+\frac{4}{3}} - \frac{\pi^{2N-\frac{2}{3}}}{4N-\frac{4}{3}})^{\frac{1}{2}} \alpha^{\frac{1}{2}} U L^{\frac{3}{2}} (\frac{\delta}{L})^{N+\frac{4}{3}}. \tag{3.39}$$

The dominant term in this estimate for large Re is clearly the ørst one. This ørst term is, remarkably, a decreasing function of L and U for $N > 2$.

4. Conclusion: Feasibility of LES. For LES with deconvolution models to be feasible for fully developed turbulence two competing restrictions on the averaging radius must simultaneously be satisøed. First, $\frac{\delta}{L}$ must be well inside the inertial range, $\frac{\delta}{L} >> \varepsilon^{-\frac{1}{4}} \nu^{\frac{3}{4}}$. Second, the models consistency error must be small: $< ||\tau|| ><< 1$. We have seem that this gives an upper bound on $\frac{\delta}{L}$which decreases as Re increases. For LES to be useful, these two constraints must be satisøed simultaneously.

To illustrate the competition between these two constraints, consider the zeroth order model ørst and suppress all constants except $\frac{\delta}{L}$ and Re. Using the consistency error bound (1.16) yields a narrow band of possible values of the averaging radius

$$C\,\mathrm{Re}^{-\frac{3}{4}} << \frac{\delta}{L} << C\,\mathrm{Re}^{-\frac{1}{2}}. \tag{4.1}$$

Thus, the extra analysis required is important for giving an accurate analytical assessment of LES. Indeed, using instead the sharper estimate $< ||\tau_0|| > \leq C(\frac{\delta}{L})^{\frac{1}{3}}$, predicts success of the zeroth order model provided

$$C\,\mathrm{Re}^{-\frac{3}{4}} << \frac{\delta}{L} << C \cdot 1. \tag{4.2}$$

For the general higher order model ($N = 1, 2, \cdots$) the dominant term in the consistency error is $< ||\tau_N|| > \leq C\,\mathrm{Re}^{\frac{3}{4}N-\frac{1}{4}} (\frac{\delta}{L})^{2N+1}$. Thus, $< ||\tau_N|| ><< 1$ provided $\frac{\delta}{L} << \mathrm{Re}^{\frac{-3N+1}{8N+4}}$. The competing feasibility conditions are thus

$$C\,\mathrm{Re}^{-\frac{3}{4}} << \frac{\delta}{L} << C\,\mathrm{Re}^{\frac{-3N+1}{8N+4}} \approx C\,\mathrm{Re}^{-\frac{3}{8}}. \tag{4.3}$$

Coupled with the observations that (i) (4.3) actually improves with increasing $N(\geq 1)$ as L, U increase, and (ii) the accuracy of the model increases dramatically as N increases, the overall analytic conclusion is that higher order models are preferable to lower order models. This observation, while surprising from the point of view of traditional error analysis, is consistent with the extensive experiments with the models. At this point we do not know if the fact that (4.3) is more restrictive than (4.2) is an essential feature of higher order models or is due to possibly improvable mechanics of our analysis.

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