Analysis of a discontinuous Galerkin and eddy viscosity method for Navier-Stokes

Songul Kaya and Béatrice Rivière

Abstract

In this paper we provide an error analysis of a subgrid scale eddy viscosity method using discontinuous polynomial approximations, for the numerical solution of the incompressible Navier-Stokes equations. Optimal continuous in time error estimates of the velocity are derived. The analysis is completed with some error estimates for two fully discrete schemes, that are first and second order in time respectively.

1 Introduction

The goal of this paper is to formulate and analyze a subgrid eddy viscosity method for solving the incompressible time-dependent Navier-Stokes equations. If the separation point between large and small scales is held fixed, the model can be viewed as a Large Eddy Simulation (LES) model. On the other hand, if the separation point is decreased as the mesh size tends to zero, the model can be viewed (and analyzed, as herein) as a numerical regularization of the Navier-Stokes equations.

For many flows in nature, capturing all the scales in a numerical simulation, is an impossible task, since the scale separation may span several orders of magnitude. Global diffusion is the traditional phenomenology to model the dispersive effects of unresolved scales on resolved scales. The traditional approach for incorporating the effects of unresolved scales on the resolved ones for the Navier-Stokes equations, utilizes eddy viscosity models. These models, first formulated by Boussinesq [6] and developed by Taylor and Prandlt [11], introduce a dissipation mechanism (Smagorinsky [30]). Standard eddy viscosity models act on all scales of motion; and their effects can be too diffusive on the coarse scales (Lewandowski [26], Iliescu and

Layton [20]). The idea of applying the eddy viscosity models only on the small scales results in the subgrid eddy viscosity method, introduced and analyzed by Guermond [14], Layton [24] and Kaya [22]. This subgrid eddy viscosity method can also be thought of as an extension to general domains and boundary conditions of the spectral vanishing viscosity idea of Maday and Tadmor [27]. Recently, Hughes and co-workers [17] proposed a Variational Multiscale Method (VMM) in which the diffusion acts only at the finest resolved scales. VMM is a promising approach in multiscale turbulence modelling. There are different choices on how to define coarse and small scales within the VMM framework. One approach is to define fluctuations via bubble functions, and means via L^2 projection (Guermond [14], Hughes [16]). Another possibility is to define fluctuations via the finest resolved scales in a hierarchy of finite element spaces, and means via elliptic or Stokes projection (Layton [24], Kaya [23], Hughes [19, 18]).

For any numerical method, the error equation arising from the Navier Stokes equations contains a convection-like term and a reaction (or stretching) term. Discontinuous Galerkin (DG) methods, first introduced in the work of Reed and Hill [29] and Lesaint and Raviart [25], are particularly efficient in controlling convective error terms. On the other hand, (generally nonlinear) eddy viscosity models are intended to give some control of the error's reaction like terms in a sense. Indeed, the exponential sensitivity of trajectories of the Navier Stokes equations (arising from reaction like term) is widely believed to be limited to the small scales. It is thus conjectured that by modelling their action on the large scales, the reaction like terms introduced exponential sensitivity will be contained.

DG methods have recently become more popular in the science and engineering community. They use piecewise polynomial functions with no continuity constraint across element interfaces. As a result, variational formulations must include jump terms across interfaces ([32]). The DG methods offers several advantages, including: (*i*) Flexibility in the design of the meshes and in the construction of trial and test spaces. (*ii*) Local conservation of mass. (*iii*) h-p adaptivity. (*iv*) Higher order local approximations. DG methods have become widely used for solving computational fluid problems, especially diffusion and pure convection problems ([3, 28]). The reader should refer to Cockburn [7] for a historical of DG methods. For the steady-state Navier-Stokes equations, a totally discontinuous finite element method is formulated in [12], while in [21], the velocity is approximated by discontinuous polynomials that are pointwise divergence-free, and the pressure by continuous polynomials.

Combining DG and eddy viscosity technique is clearly advantageous. While convective effects are accurately modelled by DG, the dispersive effects of small scales on the large scales are correctly taken into account with the eddy viscosity model. Besides, the fact that there is no constraint between the finite elements gives more freedom in choosing the appropriate the basis functions on the coarse and refined scales such as hierarchical basis functions for multiscale turbulent modelling. As an appropriate first step, we consider in this paper the combination of DG methods with a linear eddy viscosity model. We show that the errors are optimal with respect to the mesh size and depend on the Reynolds number in a reasonable fashion. The particular eddy viscosity model considered here was introduced in [24] and complete numerical analysis for Navier Stokes equation is performed in [22] where it was combined with the classical finite element method.

The outline of the paper is as follows. The model problem and notation are presented in Section 2. In Section 3, a variational formulation is introduced. Section 4 contains the continuous in time algorithm, some stability results and some error estimates. In Section 5, two fully discrete schemes are formulated and analyzed. Conclusions are given in the last section.

2 Notation and Preliminaries

We consider the stationary Navier-Stokes equations for incompressible flow as given

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f}, \text{ in } \Omega, \text{ for } 0 < t \leq T, \tag{2.1}$$

$$\nabla\cdot\boldsymbol{u} = 0, \text{ in } \Omega, \text{ for } 0 < t \leq T, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{u}_0, \text{ in } \Omega, \text{ for } t = 0, \tag{2.3}$$

$$\boldsymbol{u} = 0 \text{ on } \partial\Omega, \text{ for } 0 < t \leq T, \tag{2.4}$$

where $\boldsymbol{u}$ is the fluid velocity, p the pressure, $\boldsymbol{f}$ the external force, $\nu > 0$ the kinematic viscosity, and $\Omega \subset \mathbb{R}^2$ a bounded, simply connected domain with polygonal boundary $\partial\Omega$. We also impose the usual normalization condition on the pressure, namely that $\int_\Omega p = 0$.

Let $\mathcal{K}_h = \{E_j, j = 1, ..., N_h\}$ denote a nondegenerate triangulation of the domain Ω. Let h denote the maximum diameter of the elements E_j in $\mathcal{K}_h$. We denote the edges of $\mathcal{K}_h$ by $\{e_1, e_2, ..., e_{P_h}, e_{P_h+1}, ..., e_{M_h}\}$, where $e_k \subset \Omega$ for

$1 \le k \le P_h$ and $e_k \subset \partial\Omega$ for $P_{h+1} \le k \le M_h$. With each edge we associate a normal unit vector $\mathbf{n}_k$. For $k > P_h$, the unit vector $\mathbf{n}_k$ is taken to be outward normal to $\partial\Omega$. Let e_k be an edge shared by elements E_i and E_j with $\mathbf{n}_k$ exterior to E_i. We define the jump $[\phi]$ and average $\{\phi\}$ of a function ϕ by

$$[\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}, \quad \{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}.$$

If e belongs to the boundary $\partial\Omega$, the jump and average of ϕ coincide with its trace on e. We shall use standard notation for Sobolev spaces [1]. For $s \ge 0$ and $r \ge 1$, the classical Sobolev space on a domain $E \subset \mathbb{R}^2$ is

$$W^{s,r}(E) = \{v \in L^r(E) : \forall\, |m| \le s,\, \partial^m v \in L^r(E)\},$$

where $\partial^m v$ are the partial derivatives of v of order $|m|$. The usual norm in $W^{s,r}(E)$ is denoted by $\|\cdot\|_{s,r,E}$ and the semi norm by $|\cdot|_{s,r,E}$. The L^2 inner-product is denoted by $(\cdot,\cdot)_E$ and only by $(\cdot,\cdot)$ if $E = \Omega$. For the Hilbert space $H^s(E) = W^{s,2}(E)$, the norm is denoted by $\|\cdot\|_{s,E}$. By $H^1_0(E)$ we shall understand the subspace of $H^1(E)$ of functions that vanish on ∂E. Throughout the paper, boldface characters denote vector quantities.

For any function $\boldsymbol{\phi}$ that depends on time t and space $\boldsymbol{x}$, denote

$$\boldsymbol{\phi}(t)(\boldsymbol{x}) = \boldsymbol{\phi}(t,\boldsymbol{x}), \quad \forall t \in [0,T], \forall \boldsymbol{x} \in \Omega.$$

If Y denotes a functional space in the space variable with the norm $\|\cdot\|_Y$ and if $\boldsymbol{\phi} = \boldsymbol{\phi}(t,\boldsymbol{x})$, then for $s > 0$:

$$\|\boldsymbol{\phi}\|_{L^s(0,T;Y)} = [\int_0^T \|\boldsymbol{\phi}(t)\|_Y^s dt]^{1/s}, \quad \|\boldsymbol{\phi}\|_{L^\infty(0,T;Y)} = \max_{0\le t\le T} \|\boldsymbol{\phi}(t)\|_Y.$$

Recall that for a vector function $\boldsymbol{\phi}$, the tensor $\nabla\boldsymbol{\phi}$ is defined as $(\nabla\boldsymbol{\phi})_{i,j} = \frac{\partial\phi_i}{\partial x_j}$ and the tensor product of two tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ is defined as $\boldsymbol{T} : \boldsymbol{S} = \sum_{i,j} T_{ij} S_{ij}$. We define the following *broken* norm for positive s:

$$|||\cdot|||_s = [\sum_{j=1}^{N_h} \|\cdot\|^2_{s,E_j}]^{1/2}.$$

From [31], if $\boldsymbol{f} \in L^2(0,T;(\boldsymbol{H}^1_0)')$, there exists a solution $(\boldsymbol{u}, p)$ of (2.1)-(2.4) such that $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}^1_0(\Omega)) \cap L^2(0,T;\boldsymbol{H}^1_0)$. In addition, we will assume that $\boldsymbol{u}$ and p satisfy the following regularity properties:

- (R1) $\boldsymbol{u} \in \mathcal{C}^0(0,T;\boldsymbol{H}^1(\Omega)) \cap L^\infty(0,T;\boldsymbol{H}^2(\Omega))$
- (R2) $\boldsymbol{u}_t \in L^2(0,T;\boldsymbol{H}^1_0(\Omega))$
- (R3) $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{W}^{2,4/3}(\Omega)), \quad p \in L^\infty(0,T;W^{1,4/3}(\Omega))$.

The reader should refer to [5] for the justification of these regularity assumptions, except for the last one, that is needed here for the discontinuous Galerkin variational formulation. The following functional spaces are defined:

$$\begin{aligned}\boldsymbol{X} &= \{\boldsymbol{v} \in (L^2(\Omega))^2 : \boldsymbol{v}|_{E_j} \in \boldsymbol{W}^{2,4/3}(E_j), \quad \forall E_j \in \mathcal{K}_h\},\\ Q &= \{q \in L^2_0(\Omega) : q|_{E_j} \in W^{1,4/3}(E_j), \quad \forall E_j \in \mathcal{K}_h\},\end{aligned}$$

where $L^2_0(\Omega)$ is given by

$$L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}.$$

We associate to $(\boldsymbol{X}, Q)$ the following norms:

$$\|\boldsymbol{v}\|_X = (\|\!|\nabla \boldsymbol{v}\|\!|_0^2 + J(\boldsymbol{v},\boldsymbol{v}))^{\frac{1}{2}}, \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \quad \|q\|_Q = \|q\|_{0,\Omega}, \quad \forall q \in Q,$$

where the jump term J is defined as

$$J(\boldsymbol{u},\boldsymbol{v}) = \sum_{k=1}^{M_h} \frac{\sigma}{|e|} \int_{e_k} [\boldsymbol{u}] \cdot [\boldsymbol{v}]. \tag{2.5}$$

In this jump term, $|e|$ denotes the measure of the edge e and σ is a constant parameter that will be specified later.

Recall the following property of norm $\|\cdot\|_X$ ([12]): for each real number $p \in [2,\infty)$ there exists a constant $C(p)$ such that

$$\|\boldsymbol{v}\|_{L^p(\Omega)} \leq C(p)\|\boldsymbol{v}\|_X, \quad \forall \boldsymbol{v} \in \boldsymbol{X}. \tag{2.6}$$

For any positive integer r, the finite-dimensional subspaces are

$$\begin{aligned}\boldsymbol{X}^h &= \{\boldsymbol{v}^h \in \mathbf{X} : \boldsymbol{v}^h \in (\mathbb{P}_r(E_j))^2, \quad \forall E_j \in \mathcal{K}_h\},\\ Q^h &= \{q^h \in Q : q^h \in \mathbb{P}_{r-1}(E_j), \quad \forall E_j \in \mathcal{K}_h\}.\end{aligned}$$

We assume that for each integer $r \geq 1$, there exists an operator $R_h \in \mathcal{L}(\boldsymbol{H}^1(\Omega); \boldsymbol{X}^h)$ such that

$$\|R_h(\boldsymbol{v}) - \boldsymbol{v}\|_X \leq Ch^r |\boldsymbol{v}|_{r+1,\Omega}, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{r+1}(\Omega) \cap \boldsymbol{H}_0^1(\Omega), \tag{2.7}$$

$$|\boldsymbol{v} - \boldsymbol{R}_h(\boldsymbol{v})|_{0,E_j} \leq Ch_{E_j}^{r+1} |\boldsymbol{v}|_{r+1,\Delta_{E_j}}, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{r+1}(\Omega),\ 1 \leq j \leq N_h, \tag{2.8}$$

where Δ_{E_j} is a suitable macro element containing E_j. Note that for $r = 1, 2$ and 3, the existence of this interpolant follows from [9, 8, 10]. The bounds (2.7) and (2.8) are proved in [12] and in [13] respectively.

Also, for each integer $r \geq 1$, there is an operator $r_h \in \mathcal{L}(L_0^2(\Omega); Q_h)$ such that for any E_j in $\mathcal{K}_h$

$$\int_{E_j} z_h(r_h(q) - q) = 0, \quad \forall z_h \in \mathbb{P}_{r-1}(E_j), \quad \forall q \in L_0^2(\Omega), \tag{2.9}$$

$$\|q - r_h(q)\|_{m,E_j} \leq Ch_{E_j}^{r-m} |q|_{r,E_j}, \ \ \forall q \in H^r(\Omega) \cap L_0^2(\Omega), \quad m = 0, 1. \tag{2.10}$$

Finally, we recall some trace and inverse inequalities, that hold true on each element E in $\mathcal{K}_h$, with diameter h_E:

$$\|\boldsymbol{v}\|_{0,e} \leq C(h_E^{-1/2} \|\boldsymbol{v}\|_{0,E} + h_E^{1/2} \|\nabla \boldsymbol{v}\|_{0,E}), \quad \forall e \in \partial E, \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{2.11}$$

$$\|\nabla \boldsymbol{v}\|_{0,e} \leq C(h_E^{-1/2} \|\nabla \boldsymbol{v}\|_{0,E} + h_E^{1/2} \|\nabla^2 \boldsymbol{v}\|_{0,E}), \quad \forall e \in \partial E, \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{2.12}$$

$$\|\boldsymbol{v}\|_{L^4(e)} \leq Ch_E^{-3/4} (\|\boldsymbol{v}\|_{0,E} + h_E \|\nabla \boldsymbol{v}\|_{0,E}), \quad \forall e \in \partial E, \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{2.13}$$

$$\|\boldsymbol{v}^h\|_{0,e} \leq Ch_E^{-1/2} \|\boldsymbol{v}^h\|_{0,E}, \quad \forall e \in \partial E, \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{2.14}$$

$$\|\nabla \boldsymbol{v}^h\|_{0,e} \leq Ch_E^{-1/2} \|\nabla \boldsymbol{v}^h\|_{0,E}, \quad \forall e \in \partial E, \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{2.15}$$

$$\|\nabla \boldsymbol{v}^h\|_{0,E} \leq Ch_E^{-1} \|\boldsymbol{v}^h\|_{0,E}, \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h. \tag{2.16}$$

$$\|\boldsymbol{v}^h\|_{L^4(E)} \leq Ch_E^{-1/2} \|\boldsymbol{v}^h\|_{0,E}, \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h. \tag{2.17}$$

3 Variational Formulation

Let us first define the bilinear forms $a : \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}$ and $b : \boldsymbol{X} \times Q \to \mathbb{R}$:

$$a(\boldsymbol{v}, \boldsymbol{w}) = \sum_{j=1}^{N_h} \int_{E_j} \nabla \boldsymbol{v} : \nabla \boldsymbol{w}$$

$$- \sum_{k=1}^{M_h} \int_{e_k} (\{\nabla \boldsymbol{v}\}\mathbf{n}_k \cdot [\boldsymbol{w}] - \epsilon_0 \{\nabla \boldsymbol{w}\}\mathbf{n}_k \cdot [\boldsymbol{v}]), \tag{3.1}$$

$$b(\boldsymbol{v}, q) = - \sum_{j=1}^{N_h} \int_{E_j} q \nabla \cdot \boldsymbol{v} + \sum_{k=1}^{M_h} \int_{e_k} \{p\}[\boldsymbol{v}] \cdot \mathbf{n}_k, \tag{3.2}$$

where ϵ_0 takes the constant value 1 or -1. Throughout the paper, we will assume the following hypothesis: if $\epsilon_0 = 1$, the jump parameter σ is chosen to be equal to 1; if $\epsilon_0 = -1$, the jump parameter σ is bounded below by $\sigma_0 > 0$ and σ_0 is sufficiently large. Based on this assumption, we can easily prove the following lemma.

Lemma 3.1. *There is a constant $\kappa > 0$ such that*

$$a(\boldsymbol{v}^h, \boldsymbol{v}^h) + J(\boldsymbol{v}^h, \boldsymbol{v}^h) \geq \kappa \|\boldsymbol{v}^h\|_X^2, \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h. \tag{3.3}$$

In addition to these bilinear forms, we consider the following upwind discretization of the term $\boldsymbol{u} \cdot \nabla \boldsymbol{z}$:

$$c(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\theta}) = \sum_{j=1}^{N_h} \Big(\int_{E_j} (\boldsymbol{u} \cdot \nabla \boldsymbol{z}) \cdot \boldsymbol{\theta} + \int_{\partial E_j^-} |\{\boldsymbol{u}\} \cdot \boldsymbol{n}_{E_j}| (\boldsymbol{z}^{\text{int}} - \boldsymbol{z}^{\text{ext}}) \cdot \boldsymbol{\theta}^{int} \Big)$$

$$+ \frac{1}{2} \sum_{j=1}^{N_h} \int_{E_j} (\nabla \cdot \boldsymbol{u}) \boldsymbol{z} \cdot \boldsymbol{\theta} - \frac{1}{2} \sum_{k=1}^{M_h} \int_{e_k} [\boldsymbol{u}] \cdot \boldsymbol{n}_k \{\boldsymbol{z} \cdot \boldsymbol{\theta}\}, \tag{3.4}$$

for all $\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\theta}$ in $\mathbf{X}$ and where on each element the inflow boundary is:

$$\partial E_j^- = \{\boldsymbol{x} \in \partial E_j : \{\boldsymbol{u}\} \cdot \mathbf{n}_{E_j} < 0\},$$

and the superscript int (resp ext) refers to the trace of the function on a side of E_j coming from the interior of E_j (resp. coming from the exterior of E_j on that side). Note that the form c is not linear with respect to its first argument, but linear with respect to its second and third argument. To avoid any confusion, if necessary, in the analysis, we will explicitly write $c(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\theta}) = c_{\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\theta})$ when the inflow boundaries ∂E_j^- are defined with respect to the velocity $\{\boldsymbol{u}\}$. We finally recall the positivity of c proved in [12].

$$c(\boldsymbol{u},\boldsymbol{z},\boldsymbol{z}) \geq 0, \quad \forall \boldsymbol{u},\boldsymbol{z} \in \boldsymbol{X}. \tag{3.5}$$

With these forms, we consider a variational problem of (2.1)-(2.4): for all $t > 0$ find $\boldsymbol{u}(t) \in \boldsymbol{X}$ and $p(t) \in Q$ satisfying

$$\begin{aligned}
&(\boldsymbol{u}_t(t),\boldsymbol{v}) + \nu(a(\boldsymbol{u}(t),\boldsymbol{v}) + J(\boldsymbol{u}(t),\boldsymbol{v})) \\
&+c(\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{v}) + b(\boldsymbol{v},p(t)) = (\boldsymbol{f}(t),\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{X}, && (3.6)\\
&b(\boldsymbol{u}(t),q) = 0, \quad \forall q \in Q, && (3.7)\\
&(\boldsymbol{u}(0),\boldsymbol{v}) = (\boldsymbol{u}_0,\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{X}. && (3.8)
\end{aligned}$$

We shall now show the equivalence of the strong and weak solutions.

Lemma 3.2. *Every strong solution of (2.1)-(2.4) is also a solution of (3.6)-(3.8). Conversely, if $\boldsymbol{u} \in L^\infty(0,T;H^2(\Omega))$ and $p \in L^2(0,T;H^1(\Omega))$ are a solution of (3.6)-(3.8) then $(\boldsymbol{u},p)$ satisfies (2.1)-(2.4).*

Proof. Fix $t > 0$. Let $(\boldsymbol{u},p)$ be the solution of (2.1)-(2.4). Since $\boldsymbol{u}(t) \in \boldsymbol{H}_0^1(\Omega)$, by the trace theorem $[\boldsymbol{u}(t)]\cdot\boldsymbol{n}_k = 0$ on each edge. Also, $\nabla\cdot\boldsymbol{u}(t) = 0$, thus $\boldsymbol{u}$ satisfies (3.7). Multiplying the Navier-Stokes equation (2.1) by $\boldsymbol{v} \in \boldsymbol{X}$ and integrating over Ω yields

$$\int_\Omega (\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p)\cdot\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}.$$

We shall decompose the integrals in the above equation into element contributions and use Green's formula for each E_j.

$$\sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{u}_t\cdot\boldsymbol{v} + \nu\nabla\boldsymbol{u} : \nabla\boldsymbol{v}) - \nu\sum_{k=1}^{M_h}\int_{e_k}[\nabla\boldsymbol{u}\mathbf{n}_k\cdot\boldsymbol{v}] + \sum_{j=1}^{N_h}\int_{E_j}\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v}$$
$$-\sum_{j=1}^{N_h}\int_{E_j}p\nabla\cdot\boldsymbol{v} + \sum_{k=1}^{M_h}\int_{e_k}[p\boldsymbol{v}\cdot\mathbf{n}_k] = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}.$$

The boundary terms are rewritten as:

$$\sum_{k=1}^{M_h}\int_{e_k}[\nabla \boldsymbol{u}\boldsymbol{n}_k.\boldsymbol{v}] = \sum_{k=1}^{M_h}\int_{e_k}\{\nabla \boldsymbol{u}\}\boldsymbol{n}_k\cdot[\boldsymbol{v}] + \sum_{k=1}^{M_h}\int_{e_k}[\nabla \boldsymbol{u}]\boldsymbol{n}_k\cdot\{\boldsymbol{v}\}.$$

The first part of the lemma is then obtained because the jumps of $\boldsymbol{u}, \nabla \boldsymbol{u}\boldsymbol{n}_k$ and of p are zero almost everywhere.

Conversely, let $(\boldsymbol{u}, p)$ be a solution to (3.6)-(3.8). First, let E belong to $\mathcal{K}_h$ and choose $\boldsymbol{v} \in \mathcal{D}(E)^2$, extended by zero outside E. Then, $(\boldsymbol{u}, p)$ satisfy in the sense of distributions

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{u} = 0, \quad \text{in} \quad E. \tag{3.9}$$

Next consider $\boldsymbol{v} \in \mathcal{C}^1(\bar{E})$ such that $\boldsymbol{v} = \boldsymbol{0}$ on ∂E, extended by zero outside E, $\nabla\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on ∂E except on one side e_k. We multiply (3.9) by $\boldsymbol{v}$ and integrate by parts. We then obtain

$$\int_{e_k}\{\nabla\boldsymbol{v}\}\boldsymbol{n}_k\cdot[\boldsymbol{u}] = 0,$$

which implies that $[\boldsymbol{u}] = \boldsymbol{0}$ almost everywhere on e_k. If e_k belongs to the boundary $\partial\Omega$, this implies that $\boldsymbol{u}|_{e_k} = \boldsymbol{0}$. Thus, $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$. Finally, choose $\boldsymbol{v} \in \mathcal{C}^1(\bar{E})$, with $\boldsymbol{v} = \boldsymbol{0}$ on ∂E except on one side e_k, extended by zero outside. Integrating by parts (3.9), we have

$$\int_{e_k}(-\nu\nabla\boldsymbol{u}\boldsymbol{n}_E + p\boldsymbol{n}_E)\cdot\boldsymbol{v} = \int_{e_k}\{-\nu\nabla\boldsymbol{u}\boldsymbol{n}_E + p\boldsymbol{n}_E\}\cdot\boldsymbol{v}.$$

Since $\boldsymbol{v}$ is arbitrary, this means that the quantity $-\nu\nabla\boldsymbol{u}\boldsymbol{n}_k + p\boldsymbol{n}_k$ is continuous across e_k. Therefore, the equation (3.9) is satisfied over the entire domain Ω. The initial condition (2.3) is straightforward. □

We recall a discrete inf-sup condition and a property satisfied by R_h (see [12]).

Lemma 3.3. *There exists a positive constant β_0, independent of h such that*

$$\inf_{q^h\in Q^h}\sup_{\boldsymbol{v}^h\in\mathbf{X}^h}\frac{b(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_X\,\|q^h\|_0} \geq \beta_0. \tag{3.10}$$

Furthermore, the operator R_h satisfies:

$$b(R_h(\boldsymbol{v}) - \boldsymbol{v}, q^h) = 0, \quad \forall q^h \in Q^h, \quad \forall\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{3.11}$$

4 Finite Element Scheme

In order to subtract the artificial diffusion introduced by the eddy viscosity on the coarse grid, we consider a coarsening of the mesh $\mathcal{K}_h$, namely $\mathcal{K}_H$, such that the fine mesh $\mathcal{K}_h$ is a refinement of $\mathcal{K}_H$ (so typically $h << H$). Denote by $\boldsymbol{L}$ the space of tensors $L^2(\Omega)^{2\times 2}$ and consider the finite dimensional subspace of $\boldsymbol{L}$:

$$\boldsymbol{L}_H = \{\boldsymbol{S} \in \boldsymbol{L} : S_{ij}|_\Sigma \in \mathbb{P}_{r-1}(\Sigma),\ \forall \Sigma \in \mathcal{K}_H\}.$$

Let $P_H : \boldsymbol{L} \to \boldsymbol{L}_H$ denote the L^2 orthogonal projection on $\boldsymbol{L}_H$ and let I denote the identity mapping. Since P_H is a projection, we have the following properties

$$\|I - P_H\| \leq 1, \tag{4.1}$$

$$\|(I - P_H)\nabla \boldsymbol{v}\|_{0,\Omega} \leq CH^r |\boldsymbol{v}|_{r+1,\Omega}, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{r+1}(\Omega). \tag{4.2}$$

Throughout the paper, the variable C will denote a generic positive constant, that will take different values at different places, but will be independent of h, H, ν and ν_T. Define the following bilinear $g : \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}$:

$$g(\boldsymbol{v}, \boldsymbol{w}) = \sum_{j=1}^{N_h} \int_{E_j} (I - P_H)\nabla \boldsymbol{v} : (I - P_H)\nabla \boldsymbol{w}, \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}.$$

For all $t > 0$, we seek a discontinuous approximation $(\boldsymbol{u}^h(t), p^h(t)) \in \mathbf{X}^h \times Q^h$ such that

$$(\boldsymbol{u}_t^h(t), \boldsymbol{v}^h) + \nu(a(\boldsymbol{u}^h(t), \boldsymbol{v}^h) + J(\boldsymbol{u}^h(t), \boldsymbol{v}^h)) + \nu_T g(\boldsymbol{u}^h(t), \boldsymbol{v}^h)$$
$$+ c(\boldsymbol{u}^h(t), \boldsymbol{u}^h(t), \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h(t)) = (\boldsymbol{f}(t), \boldsymbol{v}^h), \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{4.3}$$

$$b(\boldsymbol{u}^h(t), q^h) = 0, \quad \forall q^h \in Q^h, \tag{4.4}$$

$$(\boldsymbol{u}^h(0), \boldsymbol{v}^h) = (\boldsymbol{u}_0, \boldsymbol{v}^h), \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h. \tag{4.5}$$

Lemma 4.1. *There exists a unique solution to (4.3)-(4.5).*

Proof. The equations (4.3) and (4.4) reduce to the ordinary differential system

$$\frac{d\boldsymbol{u}^h}{dt} + \nu A\boldsymbol{u}^h + B\boldsymbol{u}^h + \nu_T G\boldsymbol{u}^h = \mathbf{F}.$$

By continuity, a solution exists. To prove uniqueness, we choose $\boldsymbol{v}^h = \boldsymbol{u}^h$ in (4.3) and $q^h = p^h$ in (4.4).

$$(\boldsymbol{u}_t^h, \boldsymbol{u}^h) + \nu(a(\boldsymbol{u}^h, \boldsymbol{u}^h) + J(\boldsymbol{u}^h, \boldsymbol{u}^h)) + \nu_T \|(I - P_H)\nabla \boldsymbol{u}^h\|_0^2 \leq |(\boldsymbol{f}, \boldsymbol{u}^h)|$$

Then, applying the coercivity equation (3.3) and the generalized Cauchy-Schwarz

$$\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{u}^h\right\|_{0,\Omega}^2 + \nu\kappa\|\boldsymbol{u}^h\|_X^2 \leq \|\boldsymbol{f}\|_{L^{4/3}(\Omega)}\|\boldsymbol{u}^h\|_{L^4(\Omega)} \leq \frac{\nu\kappa}{2}\|\boldsymbol{u}^h\|_X^2 + \frac{C}{\nu\kappa}\|\boldsymbol{f}\|_{L^{4/3}(\Omega)}^2.$$

Integrating over $[0, t]$ yields:

$$\left\|\boldsymbol{u}^h(t)\right\|_{L^\infty(0,T;L^2(\Omega))}^2 + \nu\kappa\|\boldsymbol{u}^h\|_{L^2(0,T;X)}^2 \leq \left\|\boldsymbol{u}^h(0)\right\|_0^2 + \frac{C}{\nu\kappa}\|\boldsymbol{f}\|_{L^2(0,T;L^{4/3}(\Omega))}^2.$$

Since $\boldsymbol{u}^h$ is bounded in $L^\infty(0, T; L^2(\Omega))$, it is unique [4]. The existence and uniqueness of p^h is obtained from the inf-sup condition stated above. □

Remark: From a continuum mechanics point of view, it might be advantageous to consider the symmetrized velocity tensor. In this case, the bilinear form a is replaced by

$$a(\boldsymbol{v}, \boldsymbol{w}) = \sum_{j=1}^{N_h}\int_{E_j} \nabla^s \boldsymbol{v} : \nabla^s \boldsymbol{w} - \sum_{k=1}^{M_h}\int_{e_k}(\{\nabla^s \boldsymbol{v}\}\mathbf{n}_k \cdot [\boldsymbol{w}] - \epsilon_0\{\nabla^s \boldsymbol{w}\}\mathbf{n}_k \cdot [\boldsymbol{v}]),$$

where $\nabla^s \boldsymbol{v} = 0.5(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$ and the term relating the coarse and refined mesh is replaced by $\sum_{j=1}^{N_h}\int_{E_j}(I - P_H)\nabla^s \boldsymbol{u} : (I - P_H)\nabla^s \boldsymbol{v}^h$. It is easy to check that all the results proved in this paper also hold true for the symmetrized tensor formulation.

5 Semi-discrete A Priori Error Estimate

In this section, priori error estimates for the continuous in time problem, are derived. The estimates are optimal in the fine mesh size h. The effects of the coarse scale appear as higher order terms.

Theorem 5.1. *Let $(\boldsymbol{u}, p)$ be the solution of (2.1)-(2.4) satisfying R1-R3. In addition, we assume that $\boldsymbol{u}_t \in L^2(0, T; \boldsymbol{H}^{r+1}(\Omega)), \boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}^{r+1}(\Omega))$*

and $p \in L^2(0,T;\boldsymbol{H}^r(\Omega))$. *Then, the continuous in time solution* $\boldsymbol{u}_h$ *satisfies*

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{L^\infty(0,T;L^2(\Omega))} + \kappa^{1/2}\nu^{1/2}\|\boldsymbol{u}-\boldsymbol{u}^h\|_{L^2(0,T;X)} + \nu_T^{1/2}\|(I-P_H)\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\|_{L^2(0,T;L^2(\Omega))}$$
$$\leq Ce^{CT(\nu^{-1}+1)}[h^r((\nu+\nu^{-1}+\nu_T)^{1/2}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + \nu^{-1/2}|p|_{L^2(0,T;H^r(\Omega))}$$
$$+|\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}) + \nu_T^{1/2}H^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}] + Ch^r|\boldsymbol{u}_0|_{r+1,\Omega},$$

where C *is a positive constant independent of* h, H, ν *and* ν_T.

Proof. We fix $t>0$ and for simplicity, we drop the argument in t. Defining $\boldsymbol{e}^h = \boldsymbol{u}-\boldsymbol{u}^h$ and subtracting (4.3), (4.4), (4.5) from (3.6), (3.7), (3.8) respectively yields

$$(\boldsymbol{e}_t^h,\boldsymbol{v}^h) + \nu a(\boldsymbol{e}^h,\boldsymbol{v}^h) + \nu J(\boldsymbol{e}^h,\boldsymbol{v}^h) + \nu_T g(\boldsymbol{e}^h,\boldsymbol{v}^h)$$
$$+c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}^h) - c(\boldsymbol{u}^h,\boldsymbol{u}^h,\boldsymbol{v}^h) = -b(\boldsymbol{v}^h, p-p^h)$$
$$+\nu_T g(\boldsymbol{u},\boldsymbol{v}^h), \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}_h, \quad \forall t>0, \tag{5.1}$$
$$b(\boldsymbol{e}^h, q^h) = 0, \quad \forall q^h \in Q^h, \quad \forall t>0, \tag{5.2}$$
$$(\boldsymbol{e}^h(0),\boldsymbol{v}^h) = 0, \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h. \tag{5.3}$$

Decompose the error $\boldsymbol{e}^h = \boldsymbol{\eta} - \boldsymbol{\phi}^h$, where $\boldsymbol{\phi}^h = \boldsymbol{u}^h - R_h(\boldsymbol{u})$ and $\boldsymbol{\eta}$ is the interpolation error $\boldsymbol{\eta} = \boldsymbol{u} - R_h(\boldsymbol{u})$. Set $\boldsymbol{v}^h = \boldsymbol{\phi}^h$ in (5.1) and $q^h = r_h(p) - p_h$ in (5.2):

$$(\boldsymbol{\phi}_t^h,\boldsymbol{\phi}^h) + \nu a(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h) + \nu J(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h) + \nu_T g(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h)$$
$$+c_{\boldsymbol{u}^h}(\boldsymbol{u}^h,\boldsymbol{u}^h,\boldsymbol{\phi}^h) - c_{\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\phi}^h) = (\boldsymbol{\eta}_t,\boldsymbol{\phi}^h) + \nu a(\boldsymbol{\eta},\boldsymbol{\phi}^h) + \nu J(\boldsymbol{\eta},\boldsymbol{\phi}^h)$$
$$+\nu_T g(\boldsymbol{\eta},\boldsymbol{\phi}^h) + b(\boldsymbol{\phi}^h, p - r_h(p)) - \nu_T g(\boldsymbol{u},\boldsymbol{\phi}^h), \quad \forall t>0. \tag{5.4}$$

In the analysis below, we will often use Young's inequality: for any real numbers x, y, and $\delta > 0$:

$$|xy| \leq \delta x^2 + \frac{1}{4\delta}y^2.$$

We now bound the terms on the right hand-side of (5.4). The first three terms are rewritten as:

$$(\boldsymbol{\eta}_t,\boldsymbol{\phi}^h) + \nu a(\boldsymbol{\eta},\boldsymbol{\phi}^h) + \nu J(\boldsymbol{\eta},\boldsymbol{\phi}^h) = (\boldsymbol{\eta}_t,\boldsymbol{\phi}^h) + \nu \sum_{j=1}^{N_h} \int_{E_j} \nabla\boldsymbol{\eta} : \nabla\boldsymbol{\phi}^h$$
$$-\nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla\boldsymbol{\eta}\}\boldsymbol{n}_k \cdot [\boldsymbol{\phi}^h] + \nu\epsilon_0 \sum_{k=1}^{M_h} \int_{e_k} \{\nabla\boldsymbol{\phi}^h\}\boldsymbol{n}_k \cdot [\boldsymbol{\eta}] + \nu J(\boldsymbol{\eta},\boldsymbol{\phi}^h)$$
$$= S_1 + \cdots + S_5.$$

Using Cauchy-Schwarz and Young's inequalities and the approximation result (2.7), the first two terms are bounded as follows:

$$S_1 \le \|\boldsymbol{\eta}_t\|_{0,\Omega}\|\boldsymbol{\phi}^h\|_{0,\Omega} \le \frac{1}{2}\|\boldsymbol{\phi}^h\|_{0,\Omega}^2 + Ch^{2r+2}|\boldsymbol{u}_t|_{r+1,\Omega}^2,$$

$$S_2 \le 2\nu\sum_{j=1}^{N_h}\|\nabla\boldsymbol{\eta}\|_{0,E_j}\|\nabla\boldsymbol{\phi}^h\|_{0,E_j} \le \frac{\kappa\nu}{8}\|\nabla\boldsymbol{\phi}^h\|_0^2 + C\nu h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

To bound the third term, we insert the standard Lagrange interpolant of degree r, denoted by $L_h(\boldsymbol{u})$.

$$-\nu\sum_{k=1}^{M_h}\int_{e_k}\{\nabla\boldsymbol{\eta}\}\boldsymbol{n}_k\cdot[\boldsymbol{\phi}^h] = -\nu\sum_{k=1}^{M_h}\int_{e_k}\{\nabla(\boldsymbol{u}-L_h(\boldsymbol{u}))\}\boldsymbol{n}_k\cdot[\boldsymbol{\phi}^h]$$
$$-\nu\sum_{k=1}^{M_h}\int_{e_k}\{\nabla(L_h(\boldsymbol{u})-R_h(\boldsymbol{u}))\}\boldsymbol{n}_k\cdot[\boldsymbol{\phi}^h].$$

By using the inequalities (2.12) and (2.15), the definition of the jump (2.5), and the approximation results (2.7), the third term can be bounded by

$$S_3 \le \frac{\kappa\nu}{12}J(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h) + C\nu h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

Then, from the trace inequalities (2.11), (2.15) and the approximation result (2.7), we have

$$S_4 \le C\nu\Big(\sum_{k=1}^{M_h}\frac{\sigma}{|e|}\|[\boldsymbol{\phi}^h]\|_{0,e_k}^2\Big)^{1/2}\Big(\sum_{k=1}^{M_h}\frac{|e|}{\sigma}\|\{\nabla\boldsymbol{\eta}\}\|_{0,e_k}^2\Big)^{1/2}$$
$$\le \frac{\kappa\nu}{8}|\!|\!|\nabla\boldsymbol{\phi}^h|\!|\!|_0^2 + C\nu h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

The jump term is bounded by the approximation result (2.7) as follows:

$$S_5 \le \frac{\kappa\nu}{12}J(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h) + C\nu J(\boldsymbol{\eta},\boldsymbol{\eta})$$
$$\le \frac{\kappa\nu}{12}J(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h) + C\nu h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

The eddy viscosity term in the right-hand side of (5.4) is bounded from (4.1) and (2.7),

$$\nu_T g(\boldsymbol{\eta}, \boldsymbol{\phi}^h) \le \nu_T \sum_{j=1}^{N_h} \|(I - P_H)\nabla\boldsymbol{\eta}\|_{0,E_j} \|(I - P_H)\nabla\boldsymbol{\phi}^h\|_{0,E_j}$$
$$\le \frac{\nu_T}{4} \| (I - P_H)\nabla\boldsymbol{\phi}^h \|_0^2 + C\nu_T h^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2.$$

Because of (2.9), the pressure term is reduced to

$$b(\boldsymbol{\phi}^h, p - r_h(p)) = \sum_{k=1}^{M_h} \int_{e_k} \{p - r_h(p)\}[\boldsymbol{\phi}^h] \cdot \boldsymbol{n}_k,$$

which is bounded by using Cauchy-Schwarz inequality, trace inequality (2.11) and approximation result (2.10)

$$b(\boldsymbol{\phi}^h, p - r_h(p)) \le C(\|p - r_h(p)\|_0^2 + \sum_{j=1}^{N_h} h_{E_j}^2 |p - r_h(p)|_{1,E_j}^2)^{1/2} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h)^{1/2}$$
$$\le \frac{\kappa\nu}{12} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + C\frac{h^{2r}}{\nu}|p|_{r,\Omega}^2.$$

The last term on the right-hand side of (5.4), corresponding to the consistency error, is bounded using Cauchy-Schwarz inequality and the bound (4.2)

$$\nu_T g(\boldsymbol{u}, \boldsymbol{\phi}^h) \le \nu_T \sum_{j=1}^{N_h} \|(I - P_H)\nabla\boldsymbol{u}\|_{0,E_j} \|(I - P_H)\nabla\boldsymbol{\phi}^h\|_{0,E_j}$$
$$\le \frac{\nu_T}{4} \| (I - P_H)\nabla\boldsymbol{\phi}^h \|_0^2 + C\nu_T H^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2.$$

Thus far, the terms in the right-hand side of (5.4) are bounded by

$$\frac{1}{2}\|\boldsymbol{\phi}^h\|_0^2 + Ch^{2r}|\boldsymbol{u}_t|_{r+1,\Omega}^2 + C(\nu + \nu_T)h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2 + C\frac{h^{2r}}{\nu}|p|_{r,\Omega}^2$$
$$+C\nu_T H^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2 + \frac{\kappa\nu}{4}\|\boldsymbol{\phi}^h\|_X^2 + \frac{\nu_T}{2}\| (I - P_H)\nabla\boldsymbol{\phi}^h \|_0^2.$$

Consider now the nonlinear terms in (5.4). We first note that since $\boldsymbol{u}$ is continuous,

$$c_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\phi}^h) = c_{\boldsymbol{u}^h}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\phi}^h).$$

Therefore, adding and subtracting the interpolant $R_h(\boldsymbol{u})$ yields:

$$c_{\boldsymbol{u}^h}(\boldsymbol{u}^h,\boldsymbol{u}^h,\boldsymbol{\phi}^h)-c_{\boldsymbol{u}^h}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\phi}^h)=c_{\boldsymbol{u}^h}(\boldsymbol{u}^h,\boldsymbol{u}^h-R_h(\boldsymbol{u}),\boldsymbol{\phi}^h)$$
$$+c_{\boldsymbol{u}^h}(\boldsymbol{u}^h-\boldsymbol{u},R_h(\boldsymbol{u}),\boldsymbol{\phi}^h)-c_{\boldsymbol{u}^h}(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\phi}^h)$$
$$=c_{\boldsymbol{u}^h}(\boldsymbol{u}^h,\boldsymbol{\phi}^h,\boldsymbol{\phi}^h)+c_{\boldsymbol{u}^h}(\boldsymbol{\phi}^h,R_h(\boldsymbol{u}),\boldsymbol{\phi}^h)-c_{\boldsymbol{u}^h}(\boldsymbol{\eta},R_h(\boldsymbol{u}),\boldsymbol{\phi}^h)-c_{\boldsymbol{u}^h}(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\phi}^h)$$
$$=c_{\boldsymbol{u}^h}(\boldsymbol{u}^h,\boldsymbol{\phi}^h,\boldsymbol{\phi}^h)+c_{\boldsymbol{u}^h}(\boldsymbol{\phi}^h,\boldsymbol{u},\boldsymbol{\phi}^h)-c_{\boldsymbol{u}^h}(\boldsymbol{\phi}^h,\boldsymbol{\eta},\boldsymbol{\phi}^h)-c_{\boldsymbol{u}^h}(\boldsymbol{\eta},R_h(\boldsymbol{u}),\boldsymbol{\phi}^h)-c_{\boldsymbol{u}^h}(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\phi}^h).$$

To simplify the writing, we drop the subscript $\boldsymbol{u}_h$ and write $c(\cdot,\cdot,\cdot)$ for $c_{\boldsymbol{u}_h}(\cdot,\cdot,\cdot)$. From the inequality (3.5), the first term is positive. We then bound the other terms. We first note, that we can rewrite the form c as

$$c(\boldsymbol{\phi}^h,\boldsymbol{u},\boldsymbol{\phi}^h)=\sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{\phi}^h\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{\phi}^h-\frac{1}{2}b(\boldsymbol{\phi}^h,\boldsymbol{u}\cdot\boldsymbol{\phi}^h). \tag{5.5}$$

The first term, using the L^p bound (2.6), is bounded by

$$\sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{\phi}^h\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{\phi}^h\le\|\boldsymbol{\phi}^h\|_{L^4(\Omega)}\|\nabla\boldsymbol{u}\|_{L^4(\Omega)}\|\boldsymbol{\phi}^h\|_{L^2(\Omega)}$$
$$\le\frac{\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2+\frac{C}{\nu}\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}\|\boldsymbol{\phi}^h\|^2_{0,\Omega}.$$

Let $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ be the piecewise constant vectors such that

$$\boldsymbol{c}_1|_{E_j}=\frac{1}{|E_j|}\int_{E_j}\boldsymbol{u},\quad \boldsymbol{c}_2|_{E_j}=\frac{1}{|E_j|}\int_{E_j}\boldsymbol{\phi}^h,\quad 1\le j\le N_h.$$

We rewrite using (5.2) and (3.11):

$$b(\boldsymbol{\phi}^h,\boldsymbol{u}\cdot\boldsymbol{\phi}^h)=b(\boldsymbol{\phi}^h,\boldsymbol{u}\cdot\boldsymbol{\phi}^h-\boldsymbol{c}_1\cdot\boldsymbol{c}_2)=b(\boldsymbol{\phi}^h,(\boldsymbol{u}-\boldsymbol{c}_1)\cdot\boldsymbol{\phi}^h)+b(\boldsymbol{\phi}^h,\boldsymbol{c}_1\cdot(\boldsymbol{\phi}^h-\boldsymbol{c}_2)).$$

Then, expanding the first term

$$b(\boldsymbol{\phi}^h,(\boldsymbol{u}-\boldsymbol{c}_1)\cdot\boldsymbol{\phi}^h)=-\sum_{j=1}^{N_h}\int_E(\boldsymbol{u}-\boldsymbol{c}_1)\cdot\boldsymbol{\phi}^h\nabla\cdot\boldsymbol{\phi}^h$$
$$+\sum_{k=1}^{M_h}\int_{e_k}\{(\boldsymbol{u}-\boldsymbol{c}_1)\cdot\boldsymbol{\phi}^h\}[\boldsymbol{\phi}^h]\cdot\boldsymbol{n}_k=S_6+S_7.$$

The first term is bounded, for $s > 2$, using the inverse inequality (2.16) and the bound (2.6)

$$
\begin{aligned}
S_6 \le C \sum_{j=1}^{N_h} \|\boldsymbol{u} - \boldsymbol{c}_1\|_{L^s(E_j)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E_j)} \|\nabla \boldsymbol{\phi}^h\|_{L^2(E_j)} \\
\le C\|\boldsymbol{\phi}^h\|_{0,\Omega} |\boldsymbol{u}|_{W^{1,s}(\Omega)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(\Omega)} \\
\le C\|\boldsymbol{\phi}^h\|_{0,\Omega} |\boldsymbol{u}|_{W^{1,s}(\Omega)} \|\boldsymbol{\phi}^h\|_X \\
\le \frac{\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu}\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}\|\boldsymbol{\phi}^h\|_0^2.
\end{aligned}
$$

The bound for the second term is more technical. First, passing to the reference element $\hat{E}$, and using the trace inequality (2.14), we obtain

$$
S_7 \le C \sum_{k=1}^{M_h} |e_k||E|^{-1/2} \|\boldsymbol{\phi}^h\|_{0,E} \|(\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1)\cdot \hat{\boldsymbol{\phi}}^h\|_{\hat{e}}
$$

$$
\le C \sum_{k=1}^{M_h} |e_k||E|^{-1/2} \|\boldsymbol{\phi}^h\|_{0,E} (\|(\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1)\cdot \hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} + \|\hat{\nabla}((\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1)\cdot \hat{\boldsymbol{\phi}}^h)\|_{0,\hat{E}}).
$$

The L^2 term is bounded as, for $s > 2$,

$$
\begin{aligned}
\|(\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1)\cdot \hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} \le \|\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1\|_{L^s(\hat{E})} \|\hat{\boldsymbol{\phi}}^h\|_{L^{\frac{2s}{s-2}}(\hat{E})} \\
\le h|E|^{-1/s-(s-2)/(2s)} |\boldsymbol{u}|_{W^{1,s}(E)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)} \\
\le C|\boldsymbol{u}|_{W^{1,s}(E)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)}.
\end{aligned}
$$

Note for the gradient term, we write

$$
\|\hat{\nabla}((\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1)\cdot \hat{\boldsymbol{\phi}}^h)\|_{0,\hat{E}}) = \|(\hat{\nabla}\hat{\boldsymbol{u}}\cdot \hat{\boldsymbol{\phi}}^h + (\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1)\cdot \nabla \hat{\boldsymbol{\phi}}^h)\|.
$$

Let us first bound

$$
\begin{aligned}
\|\hat{\nabla}\hat{\boldsymbol{u}}\cdot \hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} \le \|\hat{\nabla}\hat{\boldsymbol{u}}\|_{L^s(\hat{E})} \|\hat{\boldsymbol{\phi}}^h\|_{L^{\frac{2s}{s-2}}(\hat{E})} \\
\le Ch|E|^{-1/s} \|\nabla \boldsymbol{u}\|_{L^s(E)} |E|^{-(s-2)/2s} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)} \\
\le C\|\nabla \boldsymbol{u}\|_{L^s(E)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)}.
\end{aligned}
$$

Now the other term is

$$\|(\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1) \cdot \hat{\nabla} \hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} \leq \|\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1\|_{L^\infty(\hat{E})} \|\hat{\nabla} \hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}}$$
$$\leq Ch\|\boldsymbol{u}\|_{L^\infty(E)} \|\nabla \boldsymbol{\phi}^h\|_{0,E}.$$

Combining all the bounds above and using (2.6), we have

$$S_7 \leq C \sum_{j=1}^{N_h} \|\boldsymbol{\phi}^h\|_{0,E_j} (|\boldsymbol{u}|_{W^{1,s}(E_j)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E_j)}$$
$$+ \|\nabla \boldsymbol{u}\|_{L^s(E_j)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E_j)} + h|\boldsymbol{u}|_{L^\infty(E_j)} \|\nabla \boldsymbol{\phi}^h\|_{L^2(E_j)})$$
$$\leq \frac{\kappa\nu}{32} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\boldsymbol{\phi}^h\|_0^2.$$

Now,

$$b(\boldsymbol{\phi}^h, \boldsymbol{c}_1 \cdot (\boldsymbol{\phi}^h - \boldsymbol{c}_2)) = -\sum_{j=1}^{N_h} \int_E \boldsymbol{c}_1 \cdot (\boldsymbol{\phi}^h - \boldsymbol{c}_2) \nabla \cdot \boldsymbol{\phi}^h$$
$$+ \sum_{k=1}^{M_h} \int_{e_k} \{\boldsymbol{c}_1 \cdot (\boldsymbol{\phi}^h - \boldsymbol{c}_2)\} [\boldsymbol{\phi}^h] \cdot \boldsymbol{n}_k = S_8 + S_9.$$

The first term is bounded by (2.16)

$$S_8 \leq C \sum_{j=1}^{N_h} \|\boldsymbol{c}_1\| \|\boldsymbol{\phi}^h - \boldsymbol{c}_2\|_{0,E_j} h^{-1} \|\boldsymbol{\phi}^h\|_{0,E_j}$$
$$\leq C \sum_{j=1}^{N_h} \|\boldsymbol{c}_1\| \|\nabla \boldsymbol{\phi}^h\|_{0,E_j} \|\boldsymbol{\phi}^h\|_{0,E_j}$$
$$\leq \frac{\kappa\nu}{64} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}^2 \|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

Similarly, the second term is bounded

$$S_9 \leq C \sum_{j=1}^{N_h} \|\boldsymbol{c}_1\| \|\nabla \boldsymbol{\phi}^h\|_{0,E_j} \|\boldsymbol{\phi}_h\|_{0,E_j}$$
$$\leq \frac{\kappa\nu}{64} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}^2 \|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

Thus,

$$c(\boldsymbol{\phi}^h,\boldsymbol{u},\boldsymbol{\phi}^h) \le \frac{5\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu}\|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

Let us now bound $c(\boldsymbol{\phi}^h,\boldsymbol{\eta},\boldsymbol{\phi}^h)$.

$$c(\boldsymbol{\phi}^h,\boldsymbol{\eta},\boldsymbol{\phi}^h) = \sum_{j=1}^{N_h}\Big(\int_{E_j}(\boldsymbol{\phi}^h\cdot\nabla\boldsymbol{\eta})\cdot\boldsymbol{\phi}^h + \int_{\partial E_j^-}|\{\boldsymbol{\phi}^h\}\cdot\boldsymbol{n}_{E_j}|(\boldsymbol{\eta}^{\mathrm{int}}-\boldsymbol{\eta}^{\mathrm{ext}})\cdot\boldsymbol{\phi}^{h,int}\Big)$$
$$-\frac{1}{2}b(\boldsymbol{\phi}^h,\boldsymbol{\eta}\cdot\boldsymbol{\phi}^h).$$

The first term is easily bounded:

$$\sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{\phi}^h\cdot\nabla\boldsymbol{\eta})\cdot\boldsymbol{\phi}^h \le \sum_{j=1}^{N_h}\|\boldsymbol{\phi}^h\|_{0,E_j}\|\boldsymbol{\phi}^h\|_{L^4(E_j)}\|\nabla\boldsymbol{\eta}\|_{L^4(E_j)}$$
$$\le \frac{\kappa\nu}{32}\|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu}\|\boldsymbol{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}^2\|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

The second term is bounded using inequalities (2.13), (2.16) and (2.6) and the estimate (2.8)

$$\sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\boldsymbol{\phi}^h\}\cdot\boldsymbol{n}_{E_j}|(\boldsymbol{\eta}^{\mathrm{int}}-\boldsymbol{\eta}^{\mathrm{ext}})\cdot\boldsymbol{\phi}^{h,int}) \le C\sum_{j=1}^{N_h}\|\boldsymbol{\phi}^h\|_{L^4(\partial E_j)}\|\boldsymbol{\eta}^h\|_{L^4(\partial E_j)}\|\boldsymbol{\phi}^h\|_{L^2(\partial E_j)}$$
$$\le C\sum_{j=1}^{N_h}h^{-3/2}h^{r+1}|\boldsymbol{u}|_{r+1,\Omega}\|\boldsymbol{\phi}^h\|_{0,\Omega}^2$$
$$\le \frac{\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2 + C\|\boldsymbol{u}\|_{L^\infty(0,T;H^{r+1}(\Omega))}^2\|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

The last term in $c(\boldsymbol{\phi}^h,\boldsymbol{\eta},\boldsymbol{\phi}^h)$ is bounded like the terms S_6, S_7, S_8 and S_9 of $c(\boldsymbol{\phi}^h,\boldsymbol{u},\boldsymbol{\phi}^h)$. The remaining nonlinear terms are bounded in a similar fashion.

$$c_{\boldsymbol{u}^h}(\boldsymbol{\eta},R_h(\boldsymbol{u}),\boldsymbol{\phi}^h) = \sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{\eta}\cdot\nabla R_h(\boldsymbol{u}))\cdot\boldsymbol{\phi}^h$$
$$+\sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\boldsymbol{\eta}\}\cdot\boldsymbol{n}_{E_j}|(R_h(\boldsymbol{u})^{\mathrm{int}}-R_h(\boldsymbol{u})^{\mathrm{ext}})\cdot\boldsymbol{\phi}^{h,\mathrm{int}} + \frac{1}{2}\sum_{j=1}^{N_h}\int_{E_j}(\nabla\cdot\boldsymbol{\eta})R_h(\boldsymbol{u})\cdot\boldsymbol{\phi}^h$$
$$-\frac{1}{2}\sum_{k=1}^{M_h}\int_{e_k}[\boldsymbol{\eta}]\cdot\boldsymbol{n}_k\{R_h(\boldsymbol{u})\cdot\boldsymbol{\phi}^h\} = S_{10}+\cdots+S_{13}$$

Using the bound (2.6) and the approximation result (2.7), we have

$$\begin{aligned} S_{10} &\le \|\boldsymbol{\eta}\|_{L^2(\Omega)} \|\nabla R_h(\boldsymbol{u})\|_{L^4(\Omega)} \|\boldsymbol{\phi}^h\|_{L^4(\Omega)} \\ &\le \frac{\kappa\nu}{64} \|\boldsymbol{\phi}^h\|_X^2 + C\|\boldsymbol{u}\|^2_{L^\infty([0,T]\times\Omega)} h^{2r} |\boldsymbol{u}|^2_{r+1,\Omega}. \end{aligned}$$

The inequalities (2.11), (2.14), (2.6) and the approximation result (2.7) yield

$$\begin{aligned} S_{11} &\le C \sum_{j=1}^{N_h} \|\boldsymbol{\eta}\|_{0,\partial E_j} \|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)} \|\boldsymbol{\phi}^h\|_{0,\partial E_j} \\ &\le C \sum_{j=1}^{N_h} h_{E_j}^{-1/2} (\|\boldsymbol{\eta}\|_{0,E_j} + h_{E_j} \|\nabla \boldsymbol{\eta}\|_{0,E_j}) h_{E_j}^{-1/2} \|\boldsymbol{\phi}^h\|_{0,E_j} \\ &\le C\|\boldsymbol{\phi}^h\|^2_{0,\Omega} + C\|\boldsymbol{u}\|^2_{L^\infty([0,T]\times\Omega)} h^{2r} |\boldsymbol{u}|^2_{r+1,\Omega}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} S_{12} &\le \sum_{j=1}^{N_h} \|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)} \|\boldsymbol{\phi}^h\|_{0,E_j} \|\nabla \cdot \boldsymbol{\eta}\|_{0,E_j} \\ &\le C\|\boldsymbol{\phi}^h\|^2_{0,\Omega} + C\|\boldsymbol{u}\|^2_{L^\infty([0,T]\times\Omega)} h^{2r} |\boldsymbol{u}|^2_{r+1,\Omega}. \end{aligned}$$

Note that S_{13} is bounded exactly like S_{11}. The other nonlinear term is bounded using (2.7) and (2.14)

$$\begin{aligned} c_{\boldsymbol{u}^h}(\boldsymbol{u}, \boldsymbol{\eta}, \boldsymbol{\phi}^h) &= \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{u} \cdot \nabla \boldsymbol{\eta}) \cdot \boldsymbol{\phi}^h + \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{u}\} \cdot \boldsymbol{n}_{E_j}| (\boldsymbol{\eta}^{\text{int}} - \boldsymbol{\eta}^{\text{ext}}) \cdot \boldsymbol{\phi}^{h,\text{int}} \\ &\le C \sum_{j=1}^{N_h} \|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)} \|\nabla \boldsymbol{\eta}\|_{0,E_j} \|\boldsymbol{\phi}^h\|_{0,E_j} + C \sum_{j=1}^{N_h} \|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)} \|\boldsymbol{\eta}\|_{0,\partial E_j} \|\boldsymbol{\phi}^h\|_{0,\partial E_j} \\ &\le C\|\boldsymbol{\phi}^h\|^2_{0,\Omega} + C\|\boldsymbol{u}\|^2_{L^\infty([0,T]\times\Omega)} h^{2r} |\boldsymbol{u}|^2_{r+1,\Omega} \end{aligned}$$

Combining all bounds above and using (3.3), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\phi}^h\|_0^2 + \frac{\kappa\nu}{2}\|\boldsymbol{\phi}^h\|_X^2 + \frac{\nu_T}{2}\|(I - P_H)\nabla\boldsymbol{\phi}^h\|_0^2 \le C(\frac{1}{\nu} + 1)\|\boldsymbol{\phi}^h\|_0^2 \\ + Ch^{2r}(\nu + \frac{1}{\nu} + \nu_T)|\boldsymbol{u}|^2_{r+1,\Omega} + C\frac{h^{2r}}{\nu}|p|^2_{r,\Omega} + Ch^{2r}|\boldsymbol{u}_t|^2_{r+1,\Omega} + C\nu_T H^{2r}|\boldsymbol{u}|^2_{r+1,\Omega}. \end{aligned}$$

Integrating over 0 and t, noting that $\|\boldsymbol{\phi}^h(0)\|_0$ is of the order h^r and using Gronwall's lemma, yields:

$$\begin{aligned}
&\|\boldsymbol{\phi}^h(t)\|_0^2 + \kappa\nu\|\boldsymbol{\phi}^h\|^2_{L^2(0,t;X)} + \nu_T\|(I-P_H)\nabla\boldsymbol{\phi}^h\|^2_{L^2(0,t;L^2(\Omega))} \\
&\leq Ce^{C(1+\nu^{-1})}h^{2r}[(\nu+\nu^{-1}+\nu_T)|\boldsymbol{u}|^2_{L^2(0,T;H^{r+1}(\Omega))} + \nu^{-1}|p|^2_{L^2(0,T;H^r(\Omega))} \\
&\quad + |\boldsymbol{u}_t|^2_{L^2(0,T;H^{r+1}(\Omega))} + \nu_T H^{2r}|\boldsymbol{u}|^2_{L^2(0,T;H^{r+1}(\Omega))}] + Ch^r|\boldsymbol{u}_0|^2_{r+1,\Omega}.
\end{aligned}$$

where the constant C is independent of ν, ν_T, h, H but depends on $\|\boldsymbol{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}$. The theorem is obtained using the approximation results (2.7), (2.8) and the following inequality:

$$\begin{aligned}
&\|\boldsymbol{u}(t)-\boldsymbol{u}^h(t)\|_0^2 + \kappa\nu\|\boldsymbol{u}(t)-\boldsymbol{u}^h(t)\|^2_{L^2(0,T;X)} + \nu_T\|(I-P_H)\nabla(\boldsymbol{u}(t)-\boldsymbol{u}^h(t))\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leq \|\boldsymbol{\phi}^h(t)\|_0^2 + \kappa\nu\|\boldsymbol{\phi}^h\|^2_{L^2(0,T;X)} + \nu_T\|(I-P_H)\nabla\boldsymbol{\phi}^h\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \|\boldsymbol{\eta}(t)\|_0^2 + \kappa\nu\|\boldsymbol{\eta}\|^2_{L^2(0,T;X)} + \nu_T\|(I-P_H)\nabla\boldsymbol{\eta}\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

□

Remark 5.1. *One of the most important property of the Theorem 5.1 is that the new method improves its robustness with respect to the Reynolds number. In most cases, error estimations of Navier Stokes equations gives a Gronwall constant that depends on the Reynolds number as $1/\nu^3$. In contrast, this approach leads to a better error estimate with a Gronwall constant depending on $1/\nu$.*

Optimal convergence rates are obtained for Theorem 5.1 if ν_T and H are appropriately chosen.

Corollary 5.1. *Assume that $\nu_T = h^\beta$ and $H = h^{1/\alpha}$. If the relation $\beta \geq 2r(\alpha-1)/\alpha$ is satisfied, then the estimate becomes*

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u}-\boldsymbol{u}^h\|_{L^2(0,T;X)} = \mathcal{O}(h^r).$$

For example, one may choose for a linear approximation the pair $(\nu_T, H) = (h, h^{1/2})$, for quadratic approximation $(\nu_T, H) = (h, h^{3/4})$ or $(\nu_T, H) = (h^2, h^{1/2})$, and for cubic approximation $(\nu_T, H) = (h, h^{5/6})$ or $(\nu_T, H) = (h^2, h^{2/3})$.

Theorem 5.2. *Under the assumptions of Theorem 5.1, and if a(.,.) is symmetric ($\epsilon_0 = -1$), the following estimate holds true*

$$\|\boldsymbol{u}_t - \boldsymbol{u}_t^h\|_{L^2(0,T;L^2(\Omega))} + \nu^{1/2}\|\boldsymbol{u} - \boldsymbol{u}^h\|_{L^\infty(0,T;X)} \le Ce^{CT\nu^{-1}}[h^r|\boldsymbol{u}_0|_{r+1,\Omega}$$
$$+h^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + h^r|\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))} + C\nu_T H^r h^{-1}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}].$$

where C and is a positive constant independent of h, H, ν and ν_T. If a(.,.) is nonsymmetric ($\epsilon_0 = 1$), the estimate is suboptimal, of order h^{r-1}.

Proof. We introduce the modified Stokes problem: for any $t > 0$, find $(\boldsymbol{u}^S(t), p^S(t)) \in \boldsymbol{X}^h \times Q^h$ such that

$$\begin{aligned}
&\nu(a(\boldsymbol{u}^S(t), \boldsymbol{v}^h) + J(\boldsymbol{u}^S(t), \boldsymbol{v}^h)) + \nu_T g(\boldsymbol{u}^S(t), \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^S(t))\\
&= \nu(a(\boldsymbol{u}(t), \boldsymbol{v}^h) + J(\boldsymbol{u}(t), \boldsymbol{v}^h)) + \nu_T g(\boldsymbol{u}(t), \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p(t)), \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h,
\end{aligned} \tag{5.6}$$

$$b(\boldsymbol{u}^S(t), q^h) = 0, \quad \forall q^h \in Q^h. \tag{5.7}$$

For any $t > 0$, there exists a unique solution to (5.6), (5.7). Furthermore, it is easy to show that the solution satisfies the error estimate:

$$\begin{aligned}
&\kappa^{1/2}\nu^{1/2}\|\boldsymbol{u}(t) - \boldsymbol{u}^S(t)\|_X + \nu_T^{1/2}\|(I - P_H)\nabla(\boldsymbol{u} - \boldsymbol{u}^S)\|_{0,\Omega}\\
&\le h^r(\nu + \nu^{-1} + \nu_T)^{1/2}|\boldsymbol{u}|_{r+1,\Omega} + \nu^{-1/2}|p|_{r,\Omega} + |\boldsymbol{u}_t|_{r+1,\Omega}) + \nu_T^{1/2} H^r|\boldsymbol{u}|_{r+1,\Omega}, \quad \forall t > 0.
\end{aligned}$$

Define $\boldsymbol{\eta} = \boldsymbol{u} - \boldsymbol{u}^S$ and $\boldsymbol{\xi} = \boldsymbol{u}^h - \boldsymbol{u}^S$, and choose the test function $\boldsymbol{v}^h = \boldsymbol{\xi}_t$. The resulting error equation is:

$$\begin{aligned}
&\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + \nu a(\boldsymbol{\xi}, \boldsymbol{\xi}_t) + \frac{\nu}{2}\frac{d}{dt}J(\boldsymbol{\xi}, \boldsymbol{\xi}) + \frac{\nu_T}{2}\frac{d}{dt}g(\boldsymbol{\xi}, \boldsymbol{\xi})\\
&= (\boldsymbol{\eta}_t, \boldsymbol{\xi}_t) - \nu_T g(\boldsymbol{u}, \boldsymbol{\xi}_t) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\xi}_t) - c(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{\xi}_t).
\end{aligned} \tag{5.8}$$

The first term in the right-hand side of (5.8) is easily bounded.

$$(\boldsymbol{\eta}_t, \boldsymbol{\xi}_t) \le \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\boldsymbol{u}_t|_{r+1,\Omega}^2.$$

The consistency error term is bounded using the inverse inequality (2.16) and the properties (4.1) and (4.2).

$$\begin{aligned}
\nu_T g(\boldsymbol{u}, \boldsymbol{\xi}_t) &\le C\nu_T H^r|\boldsymbol{u}|_{r+1,\Omega}\|\nabla\boldsymbol{\xi}_t\|_{0,\Omega}\\
&\le \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\nu_T^2 H^{2r}h^{-2}|\boldsymbol{u}|_{r+1,\Omega}^2.
\end{aligned}$$

Let us rewrite the nonlinear terms

$$c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\xi}_t)-c(\boldsymbol{u}^h,\boldsymbol{u}^h,\boldsymbol{\xi}_t)=c(\boldsymbol{u}-\boldsymbol{u}^S,\boldsymbol{u}^h,\boldsymbol{\xi}_t)-c(\boldsymbol{\xi},\boldsymbol{u}^h,\boldsymbol{\xi}_t)+c(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{u}^h,\boldsymbol{\xi}_t)$$
$$=c(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi}_t)-c(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\xi}_t)+c(\boldsymbol{\xi},\boldsymbol{u},\boldsymbol{\xi}_t)-c(\boldsymbol{\eta},\boldsymbol{u}^h,\boldsymbol{\xi}_t)+c(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\xi}_t)-c(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\xi}_t).$$

In what follows, we assume that $\boldsymbol{\xi}$ belongs to $L^\infty((0,T)\times\Omega)$. Indeed, we have the inverse inequality

$$\|\boldsymbol{\xi}\|_{L^\infty((0,T)\times\Omega)}\le C\|\boldsymbol{\xi}\|_{L^\infty((0,T),H^1(\Omega))}\le Ch^k<\infty.$$

We now consider each of the nonlinear terms. Expanding $c(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi}_t)$ results

$$\begin{aligned}c(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi}_t)=\sum_{j=1}^{N_h}\int_{E_j}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}\cdot\boldsymbol{\xi}_t+\sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\boldsymbol{\xi}\}\cdot\boldsymbol{n}_{E_j}|(\boldsymbol{\xi}^{int}-\boldsymbol{\xi}^{ext})\cdot\boldsymbol{\xi}_t^{int}\\ +\frac{1}{2}\sum_{j=1}^{N_h}(\nabla\cdot\boldsymbol{\xi})\boldsymbol{\xi}\cdot\boldsymbol{\xi}_t-\frac{1}{2}\sum_{k=1}^{M_h}\int_{e_k}[\boldsymbol{\xi}]\cdot\boldsymbol{n}_k\{\boldsymbol{\xi}\cdot\boldsymbol{\xi}_t\}\\ =S_{14}+\cdots+S_{17}.\end{aligned}$$

The first term is bounded as

$$\begin{aligned}S_{14}&\le\|\boldsymbol{\xi}\|_{L^\infty(\Omega)}\|\nabla\boldsymbol{\xi}\|_{0,\Omega}\|\boldsymbol{\xi}_t\|_{0,\Omega}\\&\le\frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2+C\|\boldsymbol{\xi}\|_X^2.\end{aligned}$$

From the definition of jump term, S_{15} is bounded

$$\begin{aligned}S_{15}&\le C\sum_{j=1}^{N_h}\|\boldsymbol{\xi}\|_{L^\infty(\Omega)}\|\boldsymbol{\xi}^{int}-\boldsymbol{\xi}^{ext}\|_{0,\partial E_j}\|\boldsymbol{\xi}_t^{int}\|_{0,\partial E_j}\\&\le C\sum_{k=1}^{M_h}\|[\boldsymbol{\xi}]\|_{0,e_k}(\frac{\sigma}{|e_k|})^{1/2-1/2}\|\boldsymbol{\xi}_t\|_{0,e_k}\\&\le\frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2+C\|\boldsymbol{\xi}\|_X^2.\end{aligned}$$

The bound for S_{16} and S_{17} is the same as S_{14}.

$$\begin{aligned}S_{16}&\le C\sum_{j=1}^{N_h}\|\boldsymbol{\xi}\|_{L^\infty(\Omega)}\|\nabla\boldsymbol{\xi}\|_{0,\Omega}\|\boldsymbol{\xi}_t\|_{0,\Omega}\\&\le\frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2+C\|\boldsymbol{\xi}\|_X^2.\end{aligned}$$

$$S_{17} \leq C \sum_{k=1}^{M_h} \|\boldsymbol{\xi}\|_{L^\infty(\Omega)} \|[\boldsymbol{\xi}]\|_{0,e_k} \|\boldsymbol{\xi}_t\|_{0,e_k}$$
$$\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C \|\boldsymbol{\xi}\|_X^2.$$

We expand the term $c(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\xi}_t)$

$$c(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\xi}_t) = \sum_{j=1}^{N_h} \int_{E_j} \boldsymbol{\xi} \cdot \nabla \boldsymbol{\eta} \cdot \boldsymbol{\xi}_t + \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{\xi}\} \cdot \boldsymbol{n}_{E_j}| (\boldsymbol{\eta}^{int} - \boldsymbol{\eta}^{ext}) \cdot \boldsymbol{\xi}_t^{int}$$
$$+ \frac{1}{2} \sum_{j=1}^{N_h} (\nabla \cdot \boldsymbol{\xi}) \boldsymbol{\eta} \cdot \boldsymbol{\xi}_t - \frac{1}{2} \sum_{k=1}^{M_h} \int_{e_k} [\boldsymbol{\xi}] \cdot \boldsymbol{n}_k \{\boldsymbol{\eta} \cdot \boldsymbol{\xi}_t\}$$
$$= S_{18} + \cdots + S_{21}.$$

By using approximation results and L^p bounds, S_{18} and S_{19} are bounded as

$$S_{18} \leq \|\boldsymbol{\xi}\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{\eta}\|_{0,\Omega} \|\boldsymbol{\xi}_t\|_{0,\Omega}$$
$$\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C h^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2,$$

$$S_{19} \leq C \sum_{j=1}^{N_h} \|\boldsymbol{\xi}\|_{L^\infty(\Omega)} \|\boldsymbol{\eta}^{int} - \boldsymbol{\eta}^{ext}\|_{0,\partial E_j} \|\boldsymbol{\xi}_t^{int}\|_{0,\partial E_j}$$
$$\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C h^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2.$$

We use the inverse inequality, and L^p bounds to bound S_{20} and S_{21}

$$S_{20} \leq C \sum_{j=1}^{N_h} \|\boldsymbol{\eta}\|_{L^4(\Omega)} \|\nabla \boldsymbol{\xi}\|_{L^2(\Omega)} \|\boldsymbol{\xi}_t\|_{L^4(\Omega)}$$
$$C h^{r+1/2} |\boldsymbol{u}|_{r+1,\Omega} \|\nabla \boldsymbol{\xi}\|_{L^2(\Omega)} h^{-1/2} \|\boldsymbol{\xi}_t\|_{0,\Omega}$$
$$\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C \|\boldsymbol{\xi}\|_X^2.$$

$$S_{21} \leq C \sum_{k=1}^{M_h} \|\boldsymbol{\xi}\|_{L^\infty(\Omega)} \|\boldsymbol{\eta}\|_{0,e_k} \|\boldsymbol{\xi}_t\|_{0,e_k}$$
$$\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C h^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2.$$

The following nonlinear term $c(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{\xi}_t)$ can be expanded as

$$c(\boldsymbol{\xi}, \boldsymbol{u}, \boldsymbol{\xi}_t) = \sum_{j=1}^{N_h} \int_{E_j} \boldsymbol{\xi} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\xi}_t + \frac{1}{2} \sum_{j=1}^{N_h} (\nabla \cdot \boldsymbol{\xi}) \boldsymbol{u} \cdot \boldsymbol{\xi}_t$$
$$- \frac{1}{2} \sum_{k=1}^{M_h} \int_{e_k} [\boldsymbol{\xi}] \cdot \boldsymbol{n}_k \{\boldsymbol{u} \cdot \boldsymbol{\xi}_t\} = S_{22} + \cdots + S_{24}.$$

Similarly, S_{22} and S_{23} are bounded by using L^p bounds, jump term and approximation results.

$$S_{22} \leq \|\boldsymbol{\xi}\|_{L^4(\Omega)} \|\nabla \boldsymbol{u}\|_{L^4(\Omega)} \|\boldsymbol{\xi}_t\|_{L^2(\Omega)}$$
$$\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C|\boldsymbol{u}|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))} \|\boldsymbol{\xi}\|_X^2.$$

$$S_{23} \leq C \sum_{j=1}^{N_h} \|\boldsymbol{u}\|_{L^\infty((0,T)\times\Omega)} \|\nabla \boldsymbol{\xi}\|_{0,\Omega} \|\boldsymbol{\xi}_t\|_{0,\Omega}$$
$$\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2.$$

$$S_{24} \leq C \sum_{k=1}^{M_h} \|\boldsymbol{u}\|_{L^\infty(\Omega)} \|[\boldsymbol{\xi}]\|_{0,e_k} \|\boldsymbol{\xi}_t\|_{0,e_k} (\frac{\sigma}{|e_k|})^{1/2-1/2}$$
$$\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2.$$

Again, we expand $c(\boldsymbol{\eta}, \boldsymbol{u}^h, \boldsymbol{\xi}_t)$

$$c(\boldsymbol{\eta}, \boldsymbol{u}^h, \boldsymbol{\xi}_t) = \sum_{j=1}^{N_h} \int_{E_j} \boldsymbol{\eta} \cdot \nabla \boldsymbol{u}^h \cdot \boldsymbol{\xi}_t + \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{\eta}\} \cdot \boldsymbol{n}_{E_j}| (\boldsymbol{u}^{h,int} - \boldsymbol{u}^{h,ext}) \cdot \boldsymbol{\xi}_t^{int}$$
$$+ \frac{1}{2} \sum_{j=1}^{N_h} (\nabla \cdot \boldsymbol{\eta}) \boldsymbol{u}^h \cdot \boldsymbol{\xi}_t - \frac{1}{2} \sum_{k=1}^{M_h} \int_{e_k} [\boldsymbol{\eta}] \cdot \boldsymbol{n}_k \{\boldsymbol{u}^h \cdot \boldsymbol{\xi}_t\}$$
$$= S_{25} + \cdots + S_{28}.$$

By using L^p bounds and approximation results the terms are bounded as

$$S_{25} \leq \|\boldsymbol{\eta}\|_{L^4(\Omega)} \|\nabla \boldsymbol{u}^h\|_{L^4(\Omega)} \|\boldsymbol{\xi}_t\|_{L^2(\Omega)}$$
$$\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2,$$

$$S_{26} \leq C \sum_{j=1}^{N_h} \|\boldsymbol{u}^h\|_{L^\infty((0,T)\times\Omega)} \|\{\boldsymbol{\eta}\}\|_{0,\partial E_j} \|\boldsymbol{\xi}_t^{int}\|_{0,\partial E_j}$$
$$\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2,$$

$$S_{27} \leq C \sum_{j=1}^{N_h} \|\boldsymbol{u}^h\|_{L^\infty((0,T)\times\Omega)} \|\nabla\boldsymbol{\eta}\|_{0,\Omega} \|\boldsymbol{\xi}_t\|_{0,\Omega}$$
$$\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

$$S_{28} \leq C \sum_{k=1}^{M_h} \|\boldsymbol{u}^h\|_{L^\infty(\Omega)} \|[\boldsymbol{\eta}]\|_{0,e_k} \|\boldsymbol{\xi}_t\|_{0,e_k}$$
$$\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

Use L^p bounds to bound $c(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\xi}_t)$

$$c(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\xi}_t) = \sum_{j=1}^{N_h} \int_{E_j} \boldsymbol{u} \cdot \nabla\boldsymbol{\xi} \cdot \boldsymbol{\xi}_t + \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{u}\} \cdot \boldsymbol{n}_{E_j}| (\boldsymbol{\xi}^{int} - \boldsymbol{\xi}^{ext}) \cdot \boldsymbol{\xi}_t^{int}$$
$$= S_{29} + S_{30}.$$

$$S_{29} \leq \|\boldsymbol{u}\|_{L^\infty((0,T)\times\Omega)} \|\nabla\boldsymbol{\xi}\|_{0,\Omega} \|\boldsymbol{\xi}_t\|_{0,\Omega}$$
$$\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2.$$

$$S_{30} \leq C \sum_{j=1}^{N_h} \|\boldsymbol{u}\|_{L^\infty((0,T)\times\Omega)} \|\boldsymbol{\xi}^{int} - \boldsymbol{\xi}^{ext}\|_{0,\partial E_j} \|\boldsymbol{\xi}_t^{int}\|_{0,\partial E_j}$$
$$\leq C \sum_{k=1}^{M_h} \|[\boldsymbol{\xi}]\|_{0,e_k} \|\boldsymbol{\xi}_t\|_{0,e_k} (\frac{\sigma}{|e_k|})^{1/2-1/2}$$
$$\leq \frac{1}{64}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2.$$

Lastly, if we expand $c(\boldsymbol{u}, \boldsymbol{\eta}, \boldsymbol{\xi}_t)$, we get

$$c(\boldsymbol{u}, \boldsymbol{\eta}, \boldsymbol{\xi}_t) = \sum_{j=1}^{N_h} \int_{E_j} \boldsymbol{u} \cdot \nabla \boldsymbol{\eta} \cdot \boldsymbol{\xi}_t + \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{u}\} \cdot \boldsymbol{n}_{E_j}| (\boldsymbol{\eta}^{int} - \boldsymbol{\eta}^{ext}) \cdot \boldsymbol{\xi}_t^{int}$$
$$= S_{31} + S_{32}.$$

These terms are bounded as following:

$$\begin{aligned} S_{31} &\leq \|\boldsymbol{u}\|_{L^\infty((0,T)\times\Omega)} \|\nabla \boldsymbol{\eta}\|_{0,\Omega} \|\boldsymbol{\xi}_t\|_{0,\Omega} \\ &\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2, \end{aligned}$$

$$\begin{aligned} S_{32} &\leq C \sum_{j=1}^{N_h} \|\boldsymbol{u}\|_{L^\infty((0,T)\times\Omega)} \|\boldsymbol{\eta}^{int} - \boldsymbol{\eta}^{ext}\|_{0,\partial E_j} \|\boldsymbol{\xi}_t^{int}\|_{0,\partial E_j} \\ &\leq \frac{1}{64} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + Ch^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2. \end{aligned}$$

Collecting all the bounds with (5.8) gives:

$$\begin{aligned} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + \nu a(\boldsymbol{\xi}, \boldsymbol{\xi}_t) + \frac{\nu}{2} \frac{d}{dt} J(\boldsymbol{\xi}, \boldsymbol{\xi}) + \frac{\nu_T}{2} \frac{d}{dt} g(\boldsymbol{\xi}, \boldsymbol{\xi}) \\ \leq \frac{1}{2} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C \|\boldsymbol{\xi}\|_X^2 + Ch^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2 \\ + Ch^{2r} |\boldsymbol{u}_t|_{r+1,\Omega}^2 + C\nu_T^2 H^{2r} h^{-2} |\boldsymbol{u}|_{r+1,\Omega}^2. \end{aligned} \tag{5.9}$$

In the case where the bilinear form a is symmetric ($\epsilon_0 = -1$), the inequality becomes

$$\begin{aligned} \frac{1}{2} \|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + \frac{\nu}{2} \frac{d}{dt} \|\boldsymbol{\xi}\|_X^2 + \frac{\nu_T}{2} \frac{d}{dt} g(\boldsymbol{\xi}, \boldsymbol{\xi}) \leq C \|\boldsymbol{\xi}\|_X^2 \\ + Ch^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2 + Ch^{2r} |\boldsymbol{u}_t|_{r+1,\Omega}^2 + C\nu_T^2 H^{2r} h^{-2} |\boldsymbol{u}|_{r+1,\Omega}^2. \end{aligned} \tag{5.10}$$

Integrating between 0 and t, and using Gronwall's lemma yields:

$$\begin{aligned} \|\boldsymbol{\xi}_t\|_{L^2(0,T;L^2(\Omega))}^2 + \nu \|\boldsymbol{\xi}\|_{L^\infty(0,T;X)}^2 + \nu_T \max_{0 \leq t \leq T} g(\boldsymbol{\xi}, \boldsymbol{\xi}) \leq Ce^{CT\nu^{-1}} [h^{2r} |\boldsymbol{u}_0|_{r+1,\Omega}^2 \\ + Ch^{2r} |\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2 + Ch^{2r} |\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}^2 + + C\nu_T^2 H^{2r} h^{-2} |\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2]. \end{aligned}$$

In the case where the bilinear form a is non-symmetric($\epsilon_0 = 1$), we rewrite (5.9) as

$$a(\boldsymbol{\xi}, \boldsymbol{\xi}_t) = \frac{1}{2}\frac{d}{dt}|||\nabla\boldsymbol{\xi}|||_0^2 - \sum_{k=1}^{M_h}\int_{e_k}\{\nabla\boldsymbol{\xi}\}\boldsymbol{n}_k\cdot[\boldsymbol{\xi}_t] + \sum_{k=1}^{M_h}\int_{e_k}\{\nabla\boldsymbol{\xi}_t\}\boldsymbol{n}_k\cdot[\boldsymbol{\xi}].$$

The bound is then suboptimal: $\mathcal{O}(h^{r-1})$. □

We now derive an error estimate for the pressure.

Theorem 5.3. *Assume that a(.,.) is symmetric ($\epsilon_0 = -1$) and $\nu \le 1$. In addition, we assume that $\boldsymbol{u} \in L^2(0,T;H^{r+1})$, $\boldsymbol{u}_t \in L^2(0,T;H^{r+1})$ and $p \in L^2(0,T;H^r)$. Then, the solution p^h satisfies the following error estimate*

$$\begin{aligned}
\|p^h - r_h(p)\|_{L^2(0,T;L^2(\Omega))} &\le Ce^{CT\nu^{-1}}[\nu h^r|\boldsymbol{u}_0|_{r+1,\Omega} \\
+\nu h^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} &+ \nu h^r|\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))} + C\nu\nu_T H^r h^{-1}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}] \\
+C\nu^{1/2}h^r|\boldsymbol{u}_0|_{r+1,\Omega} &+ C\nu h^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + C\nu h^r|p|_{L^2(0,T;H^r(\Omega))} \\
&+C\nu_T H^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} \\
+Ce^{CT(\nu^{-1}+1)}[h^r((\nu+\nu^{-1}+\nu_T)^{1/2}&|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + \nu^{-1/2}|p|_{L^2(0,T;H^r(\Omega))} \\
+|\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}) &+ \nu_T^{1/2}H^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}] + Ch^r|\boldsymbol{u}_0|_{r+1,\Omega}.
\end{aligned}$$

where C is independent of h, H, ν and ν_T. Again, if $\epsilon_0 = 1$, the estimate is suboptimal.

Proof. The error equation can be written for all $\boldsymbol{v}^h$ in $\boldsymbol{X}^h$:

$$\begin{aligned}
-b(\boldsymbol{v}^h, p^h - r_h(p)) &= (\boldsymbol{u}_t^h - \boldsymbol{u}_t, \boldsymbol{v}^h) + \nu a(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{v}^h) + \nu J(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{v}^h) \\
+\nu_T g(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{v}^h) &+ c(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) - c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}^h) + \nu_T g(\boldsymbol{u}, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p - r_h(p)).
\end{aligned}$$

From the inf-sup condition (3.10), there is $\boldsymbol{v}^h \in \boldsymbol{X}^h$ such that

$$b(\boldsymbol{v}^h, p^h - r_h(p)) = -\|p^h - r_h(p)\|_0^2, \quad \|\boldsymbol{v}^h\|_X \le \frac{1}{\beta_0}\|p^h - r_h(p)\|_{0,\Omega}.$$

Thus, we have

$$\|p^h - r_h(p)\|_{0,\Omega}^2 = (\boldsymbol{u}_t^h - \boldsymbol{u}_t, \boldsymbol{v}^h) + \nu\sum_{j=1}^{N_h}\int_{E_j}\nabla(\boldsymbol{u}^h - \boldsymbol{u}) : \nabla\boldsymbol{v}^h$$

$$-\nu\sum_{k=1}^{M_h}\int_{e_k}\{\nabla(\boldsymbol{u}^h - \boldsymbol{u})\}\boldsymbol{n}_k\cdot[\boldsymbol{v}^h] + \nu\epsilon_0\sum_{k=1}^{M_h}\int_{e_k}\{\nabla\boldsymbol{v}^h\}\boldsymbol{n}_k\cdot[\boldsymbol{u}^h - \boldsymbol{u}] + \nu J(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{v}^h)$$

$$+\nu_T g(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{v}^h) + c(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) - c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}^h) + \nu_T g(\boldsymbol{u}, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p - r_h(p)).$$

All the terms above can be handled as in Theorem 5.1. The resulting inequality is

$$\|p^h - r_h(p)\|_{0,\Omega}^2 \leq C\nu^2\|\boldsymbol{u}_t^h - \boldsymbol{u}_t\|_{0,\Omega}^2 + C\nu^2\|\boldsymbol{u}^h - \boldsymbol{u}\|_X^2 + C\nu^2 h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2 + C\nu^2 h^{2r}|p|_{r,\Omega}^2$$
$$+C\nu_T^2 H^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2 + C\nu_T^2 g(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{u}) + C\|\boldsymbol{u}^h - \boldsymbol{u}\|_{0,\Omega}^2.$$

We now integrate between 0 and T, and use Theorem 5.1 and Theorem 5.2 to conclude. □

6 Fully discrete scheme

In this section, we formulate two fully discrete finite element schemes for the discontinuous eddy viscosity method. Let Δt denote the time step, let $M = T/\Delta t$ and let $0 = t_0 < t_1 < \cdots < t_M = T$ be a subdivision of the interval $(0,T)$. We denote the function ϕ evaluated at the time t_m by ϕ_m and the average of ϕ at two successive time levels by $\phi_{m+\frac{1}{2}} = \frac{1}{2}(\phi_m + \phi_{m+1})$.

Scheme 1: Given $\boldsymbol{u}_0^h$, find $(\boldsymbol{u}_m^h)_{m\geq 1}$ in $\boldsymbol{X}^h$ and $(p_m^h)_{m\geq 1}$ in Q^h such that

$$\frac{1}{\Delta t}(\boldsymbol{u}_{m+1}^h - \boldsymbol{u}_m^h, \boldsymbol{v}^h) + \nu(a(\boldsymbol{u}_{m+1}^h, \boldsymbol{v}^h) + J(\boldsymbol{u}_{m+1}^h, \boldsymbol{v}^h)) + c(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{v}^h)$$
$$+\nu_T g(\boldsymbol{u}_{m+1}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p_{m+1}^h) = (\boldsymbol{f}_{m+1}, \boldsymbol{v}^h), \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \qquad (6.1)$$
$$b(\boldsymbol{u}_{m+1}^h, q^h) = 0, \quad \forall q^h \in Q^h. \qquad (6.2)$$

Scheme 2: Given $\tilde{\boldsymbol{u}}_0^h, \tilde{\boldsymbol{u}}_1^h, \tilde{p}_1^h$, find $(\tilde{\boldsymbol{u}}_m^h)_{m\geq 2}$ in $\boldsymbol{X}^h$ and $(\tilde{p}_m^h)_{m\geq 2}$ in Q^h such that

$$\frac{1}{\Delta t}(\tilde{\boldsymbol{u}}_{m+1}^h - \tilde{\boldsymbol{u}}_m^h, \boldsymbol{v}^h) + \nu(a(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \boldsymbol{v}^h) + J(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \boldsymbol{v}^h)) + c(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \boldsymbol{v}^h)$$
$$+\nu_T g(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, \tilde{p}_{m+\frac{1}{2}}^h) = (\boldsymbol{f}_{m+\frac{1}{2}}, \boldsymbol{v}^h), \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \qquad (6.3)$$
$$b(\tilde{\boldsymbol{u}}_{m+1}^h, q^h) = 0, \quad \forall q^h \in Q^h. \qquad (6.4)$$

For both schemes, the initial velocity is defined to be the L^2 projection of $\boldsymbol{u}_0$. Scheme 1 is based on a backward Euler discretization. Scheme 2 is based on a Crank-Nicolson discretization, and requires the velocity and pressure at the first step. The approximations $\tilde{\boldsymbol{u}}_1^h$ and $\tilde{p}_1^h$ can be obtained by a first order scheme (see [2]). We will show that Scheme 1 is first order in time, and Scheme 2 second order in time. First, we prove the stability of the schemes.

Lemma 6.1. *The solution* $(\boldsymbol{u}_m^h)_m$ *of (6.1),(6.2) remains bounded in the following sense*

$$\|\boldsymbol{u}_m^h\|_{0,\Omega}^2 \leq K, \quad m = 0, \dots, M,$$
$$\Delta t \sum_{m=0}^{M-1} \|\boldsymbol{u}_{m+1}^h\|_X^2 \leq \frac{K}{2\nu},$$
$$\Delta t \sum_{m=0}^{M-1} |||(I - P_H)\nabla \boldsymbol{u}_{m+1}^h|||_0^2 \leq \frac{K}{2\nu_T},$$

where $K = \|\boldsymbol{u}_0\|_{0,\Omega}^2 + \|\boldsymbol{f}\|_{L^2([0,T]\times\Omega)}^2$.

The solution $(\tilde{\boldsymbol{u}}_m^h)_m$ *of (6.3),(6.4) remains bounded in the following sense*

$$\|\tilde{\boldsymbol{u}}_m^h\|_{0,\Omega}^2 \leq \tilde{K}, \quad m = 0, \dots, M,$$
$$\Delta t \sum_{m=0}^{M-1} \|\tilde{\boldsymbol{u}}_{m+1}^h\|_X^2 \leq \frac{\tilde{K}}{2\nu},$$
$$\Delta t \sum_{m=0}^{M-1} \|(I - P_H)\nabla \tilde{\boldsymbol{u}}_{m+1}^h\|_{0,\Omega}^2 \leq \frac{\tilde{K}}{2\nu_T},$$

where $\tilde{K} = \|\boldsymbol{u}_0\|_{0,\Omega}^2 + 2\|\boldsymbol{f}\|_{L^2([0,T]\times\Omega)}^2$.

Proof. Choose $\boldsymbol{v}^h = \boldsymbol{u}_{m+1}^h$ in (6.1) and $q^h = p_{m+1}^h$ in (6.2). From the positivity of c and (3.3), we have

$$\frac{1}{2\Delta t}(\|\boldsymbol{u}_{m+1}^h\|_{0,\Omega}^2 - \|\boldsymbol{u}_m^h\|_{0,\Omega}^2) + \kappa\nu\|\boldsymbol{u}_{m+1}^h\|_X^2$$
$$+\nu_T|||(I - P_H)\nabla \boldsymbol{u}_{m+1}^h|||_0^2 \leq \frac{1}{2}\|\boldsymbol{f}_{m+1}\|_{0,\Omega}^2 + \frac{1}{2}\|\boldsymbol{u}_{m+1}^h\|_{0,\Omega}^2.$$

Multiply by $2\Delta t$ and sum over m:

$$\|\boldsymbol{u}_m^h\|_{0,\Omega}^2 - \|\boldsymbol{u}_0^h\|_{0,\Omega}^2 + 2\kappa\nu\Delta t \sum_{j=0}^{m-1} \|\boldsymbol{u}_{j+1}^h\|_X^2 + 2\nu_T\Delta t \sum_{j=0}^{m-1} |||(I - P_H)\nabla \boldsymbol{u}_{j+1}^h|||_0^2$$
$$\leq \Delta t \sum_{j=0}^{m-1} \|\boldsymbol{f}_{j+1}\|_{0,\Omega}^2 + \Delta t \sum_{j=0}^{m-1} \|\boldsymbol{u}_{j+1}^h\|_{0,\Omega}^2.$$

The result is obtained by using a discrete version of Gronwall's lemma [15] and the fact that $\|\boldsymbol{u}_0^h\|_{0,\Omega} \leq \|\boldsymbol{u}_0\|_{0,\Omega}$.

For Scheme 2, the proof is similar. Choose $\boldsymbol{v}^h = \tilde{\boldsymbol{u}}_{m+\frac{1}{2}}$ in (6.3) and $q^h = \tilde{p}^h_{m+\frac{1}{2}}$ in (6.4).

$$\frac{1}{2\Delta t}(\|\tilde{\boldsymbol{u}}^h_{m+1}\|^2_{0,\Omega} - \|\tilde{\boldsymbol{u}}^h_m\|^2_{0,\Omega}) + \kappa\nu\|\tilde{\boldsymbol{u}}^h_{m+\frac{1}{2}}\|^2_X$$
$$+\nu_T\|\!|(I-P_H)\nabla\tilde{\boldsymbol{u}}^h_{m+\frac{1}{2}}\|\!|^2_0 \leq \frac{1}{2}\|\boldsymbol{f}_{m+\frac{1}{2}}\|^2_{0,\Omega} + \frac{1}{2}(\|\tilde{\boldsymbol{u}}^h_{m+1}\|^2_{0,\Omega} + \|\tilde{\boldsymbol{u}}^h_m\|^2_{0,\Omega}).$$

The rest of the proof follows as above. □

Theorem 6.1. *Under the assumptions of Theorem 5.1 and if $\boldsymbol{u}_t$ and $\boldsymbol{u}_{tt}$ belong to $L^\infty(0,T;L^2(\Omega))$, there is a constant C independent of h, H, ν and ν_T such that*

$$\max_{m=0,\dots,M}\|\boldsymbol{u}_m - \boldsymbol{u}^h_m\|_{0,\Omega} + (\nu\kappa\Delta t\sum_{m=0}^{M-1}\|\boldsymbol{u}_{m+1} - \boldsymbol{u}^h_{m+1}\|^2_X)^{1/2}$$
$$+(\nu_T\Delta t\sum_{m=0}^{M}\|\!|(I-P_H)(\nabla\boldsymbol{u}_{m+1} - \boldsymbol{u}^h_{m+1})\|\!|^2_0)^{1/2} \leq$$
$$Ce^{CT\nu^{-1}}[h^r(\nu+\nu^{-1}+\nu_T)^{1/2}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + \nu_T^{1/2}H^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}$$
$$+\nu^{-1/2}\Delta t(\|\boldsymbol{u}_t\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}) + h^r\nu^{-1/2}|p|_{L^2(0,T;H^r(\Omega))}]$$
$$+ Ch^r|\boldsymbol{u}_0|_{r+1,\Omega}.$$

Proof. As in the continuous case, we set $\boldsymbol{e}_m = \boldsymbol{u}_m - \boldsymbol{u}^h_m$. We subtract to (6.1) and (6.2) the equations (4.3) and (4.4) evaluated at time $t = t_{m+1}$.

$$(\boldsymbol{u}_t(t_{m+1}),\boldsymbol{v}^h) - \frac{1}{\Delta t}(\boldsymbol{u}^h_{m+1} - \boldsymbol{u}^h_m,\boldsymbol{v}^h) + \nu[a(\boldsymbol{e}_{m+1},\boldsymbol{v}^h) + J(\boldsymbol{e}_{m+1},\boldsymbol{v}^h)]$$
$$+\nu_T g(\boldsymbol{e}_{m+1},\boldsymbol{v}^h) + c(\boldsymbol{u}_{m+1},\boldsymbol{u}_{m+1},\boldsymbol{v}^h) - c(\boldsymbol{u}^h_m,\boldsymbol{u}^h_{m+1},\boldsymbol{v}^h)$$
$$+b(\boldsymbol{v}^h, p_{m+1} - p^h_{m+1}) = \nu_T g(\boldsymbol{u}_{m+1},\boldsymbol{v}^h), \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{6.5}$$
$$b(\boldsymbol{e}_{m+1}, q^h) = 0, \quad \forall q^h \in Q^h. \tag{6.6}$$

Define $\boldsymbol{\phi}_m = \boldsymbol{u}^h_m - (R_h(\boldsymbol{u}))_m$, $\boldsymbol{\eta}_m = \boldsymbol{u}_m - (R_h(\boldsymbol{u}))_m$. Choose $\boldsymbol{v}^h = \boldsymbol{\phi}_{m+1}$ in (6.5) and $q^h = p^h_{m+1}$ in (6.6). Adding and subtracting the interpolant and using (3.3) yields the following error equation:

$$\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \nu\kappa\|\boldsymbol{\phi}_{m+1}\|_X^2 + \nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!|\!|_0^2$$
$$+c(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{u}_{m+1}, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}) + b(\boldsymbol{\phi}_{m+1}, p_{m+1}^h - p_{m+1})$$
$$\leq \|\frac{\partial \boldsymbol{u}}{\partial t}(t_{m+1}) - \frac{1}{\Delta t}(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m)\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega} + \frac{1}{\Delta t}\|\boldsymbol{\eta}_{m+1} - \boldsymbol{\eta}_m\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}$$
$$+\nu|a(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) + J(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})| + \nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{\eta}_{m+1}|\!|\!|_0|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!|\!|_0$$
$$+\nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{u}_{m+1}|\!|\!|_0|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!|\!|_0.$$

We rewrite the nonlinear terms:

$$c_{\boldsymbol{u}_m^h}(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - c_{\boldsymbol{u}_{m+1}}(\boldsymbol{u}_{m+1}, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})$$
$$= c_{\boldsymbol{u}_m^h}(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - c_{\boldsymbol{u}_m^h}(\boldsymbol{u}_{m+1}, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}).$$

We now drop the subscript $\boldsymbol{u}_m^h$.

$$c_{\boldsymbol{u}_m^h}(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - c_{\boldsymbol{u}_m^h}(\boldsymbol{u}_{m+1}, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}) =$$
$$c(\boldsymbol{u}_m^h, \boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1}) + c(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^I, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{u}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})$$
$$= c(\boldsymbol{u}_m^h, \boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1}) + c(\boldsymbol{\phi}_m, \boldsymbol{u}_{m+1}^I, \boldsymbol{\phi}_{m+1}) + c(\boldsymbol{u}_m^I, \boldsymbol{u}_{m+1}^I, \boldsymbol{\phi}_{m+1})$$
$$-c(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{u}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})$$
$$= c(\boldsymbol{u}_m^h, \boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1}) + c(\boldsymbol{\phi}_m, \boldsymbol{u}_{m+1}^I, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{\eta}_m, \boldsymbol{u}_{m+1}^I, \boldsymbol{\phi}_{m+1}) + c(\boldsymbol{u}_m, \boldsymbol{u}_{m+1}^I, \boldsymbol{\phi}_{m+1})$$
$$-c(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{u}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})$$
$$= c(\boldsymbol{u}_m^h, \boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{\phi}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) + c(\boldsymbol{\phi}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{\eta}_m, \boldsymbol{u}_{m+1}^I, \boldsymbol{\phi}_{m+1})$$
$$-c(\boldsymbol{u}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}).$$

Thus, we rewrite the error equation as

$$\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \nu\kappa\|\boldsymbol{\phi}_{m+1}\|_X^2 + \nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!|\!|_0^2$$
$$+c(\boldsymbol{u}_m^h, \boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1}) \leq |c(\boldsymbol{\phi}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})|$$
$$+|c(\boldsymbol{\phi}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})| + |c(\boldsymbol{\eta}_m, \boldsymbol{u}_{m+1}^I, \boldsymbol{\phi}_{m+1})|$$
$$+|c(\boldsymbol{u}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})|$$
$$+|c(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})| + |b(\boldsymbol{\phi}_{m+1}, p_{m+1}^h - p_{m+1})|$$
$$+\|\frac{\partial \boldsymbol{u}}{\partial t}(t_{m+1}) - \frac{1}{\Delta t}(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m)\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega} + \frac{1}{\Delta t}\|\boldsymbol{\eta}_{m+1} - \boldsymbol{\eta}_m\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}$$
$$+\nu|a(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) + J(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})| + \nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{\eta}_{m+1}|\!|\!|_0|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!|\!|_0$$
$$+\nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{u}_{m+1}|\!|\!|_0|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!|\!|_0 \leq |T_0| + \cdots + |T_{10}|.$$

We want to bound the terms $T_0, T_2....T_{10}$. We can rewrite the first term as

$$T_0 = \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\phi}_m \cdot \nabla \boldsymbol{\eta}_{m+1}) \cdot \boldsymbol{\phi}_{m+1} + \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{\phi}_m\} \cdot \boldsymbol{n}_{E_j}| (\boldsymbol{\eta}_{m+1}^{\text{int}} - \boldsymbol{\eta}_{m+1}^{\text{ext}}) \cdot \boldsymbol{\phi}_{m+1}^{int}$$

$$+ \frac{1}{2} \sum_{j=1}^{N_h} \int_{E_j} (\nabla \cdot \boldsymbol{\phi}_m) \boldsymbol{\eta}_{m+1} \cdot \boldsymbol{\phi}_{m+1} - \frac{1}{2} \sum_{k=1}^{P_h} \int_{e_k} [\boldsymbol{\phi}_m] \cdot \boldsymbol{n}_k \{\boldsymbol{\eta}_{m+1} \cdot \boldsymbol{\phi}_{m+1}\}$$

$$= T_{01} + \cdots + T_{04}.$$

By using Cauchy-Schwarz, Young's inequality and L^p bound (2.6), we have:

$$T_{01} \leq \sum_{j=1}^{N_h} \|\boldsymbol{\phi}_{m+1}\|_{L^4(E_j)} \|\boldsymbol{\phi}_m\|_{L^2(E_j)} \|\nabla \boldsymbol{\eta}_{m+1}\|_{L^4(E_j)}$$

$$\leq C \|\boldsymbol{\phi}_{m+1}\|_{L^4(\Omega)} \|\boldsymbol{\phi}_m\|_{L^2(\Omega)} \|\nabla \boldsymbol{\eta}_{m+1}\|_{L^4(\Omega)}$$

$$\leq \frac{\kappa \nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C \nu^{-1} \|\boldsymbol{\phi}_m\|_{0,\Omega}^2 \|\boldsymbol{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}^2.$$

The bound for T_{02} is obtained by using the trace inequalities for L^2 (2.14), (2.13), and the approximation results (2.7), (2.8).

$$T_{02} \leq C \sum_{j=1}^{N_h} \|\boldsymbol{\phi}_m\|_{L^2(\partial E_j)} \|\boldsymbol{\eta}_{m+1}\|_{L^4(\partial E_j)} \|\boldsymbol{\phi}_{m+1}\|_{L^4(\partial E_j)}$$

$$\leq C \sum_{j=1}^{N_h} h_{E_j}^{-1/2} \|\boldsymbol{\phi}_m\|_{0,E_j} h_{E_j}^{-3/2} (\|\boldsymbol{\eta}_{m+1}\|_{0,E_j} + h_{E_j} \|\nabla \boldsymbol{\eta}_{m+1}\|_{0,E_j}) (\|\boldsymbol{\phi}_{m+1}\|_{0,E_j} + h_{E_j} \|\nabla \boldsymbol{\phi}_{m+1}\|_{0,E_j})$$

$$\leq C \sum_{j=1}^{N_h} h_{E_j}^{-2} h_{E_j}^{r+1} |\boldsymbol{u}_{m+1}|_{r+1,E_j} (\|\boldsymbol{\phi}_{m+1}\|_{0,E_j} + h_{E_j} \|\nabla \boldsymbol{\phi}_{m+1}\|_{0,E_j}) \|\boldsymbol{\phi}_m\|_{0,E_j}$$

$$\leq \frac{\kappa \nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C \nu^{-1} \|\boldsymbol{u}\|_{L^\infty(0,T;H^{r+1}(\Omega))}^2 \|\boldsymbol{\phi}_m\|_{0,\Omega}^2.$$

Similarly, the inverse inequalities (2.16), (2.17), the bound (2.6) and the approximation result (2.8) yields

$$
\begin{aligned}
T_{03} &\le C\sum_{j=1}^{N_h} h_{E_j}^{-1}\|\boldsymbol{\phi}_m\|_{L^2(E_j)}\|\boldsymbol{\eta}_{m+1}\|_{L^4(E_j)}\|\boldsymbol{\phi}_{m+1}\|_{L^4(E_j)}\\
&\le Ch^{-1}\|\boldsymbol{\phi}_m\|_{L^2(\Omega)}\|\boldsymbol{\eta}_{m+1}\|_{L^4(\Omega)}\|\boldsymbol{\phi}_{m+1}\|_{L^4(\Omega)}\\
&\le Ch^{-3/2}\|\boldsymbol{\phi}_m\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_X\|\boldsymbol{\eta}_{m+1}\|_{0,\Omega}\\
&\le Ch^{r-1/2}\|\boldsymbol{\phi}_m\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_X|\boldsymbol{u}_{m+1}|_{r+1,\Omega}\\
&\le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\|\boldsymbol{u}\|^2_{L^\infty(0,T;H^{r+1}(\Omega))}\|\boldsymbol{\phi}_m\|^2_{0,\Omega}.
\end{aligned}
$$

The bound for the term T_{04} is the same as for T_{02}.

$$
\begin{aligned}
T_{04} &\le C\sum_{k=1}^{M_h}\|\boldsymbol{\phi}_m\|_{L^2(e_k)}\|\boldsymbol{\eta}_{m+1}\|_{L^4(e_k)}\|\boldsymbol{\phi}_{m+1}\|_{L^4(e_k)}\\
&\le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\|\boldsymbol{u}\|^2_{L^\infty(0,T;H^{r+1}(\Omega))}\|\boldsymbol{\phi}_m\|^2_{0,\Omega}.
\end{aligned}
$$

The term T_1 is bounded exactly like the term (5.5) in the proof of Theorem 5.1. Here, the constant vectors are

$$
\boldsymbol{c}_1 = \frac{1}{|E_j|}\int_{E_j}\boldsymbol{u}_{m+1}, \quad \boldsymbol{c}_2 = \frac{1}{|E_j|}\int_{E_j}\boldsymbol{\phi}_{m+1}.
$$

Then, T_1 can be rewritten as:

$$
\begin{aligned}
T_1 = \sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{\phi}_m\cdot\nabla\boldsymbol{u}_{m+1})\cdot\boldsymbol{\phi}_{m+1}\\
-\frac{1}{2}b(\boldsymbol{\phi}_m,(\boldsymbol{u}_{m+1}-\boldsymbol{c}_1)\cdot\boldsymbol{\phi}_{m+1}) - \frac{1}{2}b(\boldsymbol{\phi}_m,\boldsymbol{c}_1\cdot(\boldsymbol{\phi}_{m+1}-\boldsymbol{c}_2))\\
\le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\|\boldsymbol{\phi}_m\|^2_{0,\Omega}.
\end{aligned}
$$

Expanding T_2, we obtain:

$$
\begin{aligned}
T_2 = \sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{\eta}_m\cdot\nabla\boldsymbol{u}^I_{m+1})\cdot\boldsymbol{\phi}_{m+1} + \sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\boldsymbol{\eta}_m\}\cdot\boldsymbol{n}_{E_j}|(\boldsymbol{u}^{I,\mathrm{int}}_{m+1}-\boldsymbol{u}^{I,\mathrm{ext}}_{m+1})\cdot\boldsymbol{\phi}^{int}_{m+1}\\
+\frac{1}{2}\sum_{j=1}^{N_h}\int_{E_j}(\nabla\cdot\boldsymbol{\eta}_m)\boldsymbol{u}^I_{m+1}\cdot\boldsymbol{\phi}_{m+1} - \frac{1}{2}\sum_{k=1}^{P_h}\int_{e_k}[\boldsymbol{\eta}_m]\cdot\boldsymbol{n}_k\{\boldsymbol{u}^I_{m+1}\cdot\boldsymbol{\phi}_{m+1}\}\\
= T_{21}+\cdots+T_{24}.
\end{aligned}
$$

The bound for T_{21} is obtained using (2.6) and (2.8):

$$
\begin{aligned}
T_{21} &\leq \|\boldsymbol{\eta}_m\|_{0,\Omega}\|\nabla \boldsymbol{u}^I_{m+1}\|_{L^4(\Omega)}\|\boldsymbol{\phi}_{m+1}\|_{L^4(\Omega)} \\
&\leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|^2_X + C\nu^{-1}h^{2r}\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}|\boldsymbol{u}_m|^2_{r+1,\Omega}.
\end{aligned}
$$

Similarly for the term T_{22}, the inequalities (2.7) and (2.14) give

$$
\begin{aligned}
T_{22} &\leq C\sum_{j=1}^{N_h}\|\boldsymbol{\eta}_m\|_{L^2(\partial E_j)}\|\boldsymbol{u}^I_{m+1}\|_{L^\infty(\Omega)}\|\boldsymbol{\phi}_{m+1}\|_{L^2(\partial E_j)} \\
&\leq C\sum_{j=1}^{N_h} h^r|\boldsymbol{u}_m|_{r+1,E_j}\|\boldsymbol{u}_{m+1}\|_{L^\infty(\Omega)}\|\boldsymbol{\phi}_{m+1}\|_{0,E_j} \\
&\leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|^2_X + C\nu^{-1}h^{2r}\|\boldsymbol{u}\|^2_{L^\infty([0,T]\times\Omega)}|\boldsymbol{u}_m|^2_{r+1,\Omega}.
\end{aligned}
$$

The estimate of T_{23} is obtained by using a bound on interpolant, Cauchy-Schwarz inequality, the approximation result (2.7), Young's inequality and L^p bound (2.6).

$$
\begin{aligned}
T_{23} &\leq C\sum_{j=1}^{N_h}\|\nabla\cdot\boldsymbol{\eta}_m\|_{0,E_j}\|\boldsymbol{u}^I_{m+1}\|_{L^\infty(\Omega)}\|\boldsymbol{\phi}_{m+1}\|_{0,E_j} \\
&\leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|^2_X + C\nu^{-1}h^{2r}\|\boldsymbol{u}\|^2_{L^\infty([0,T]\times\Omega)}|\boldsymbol{u}_m|^2_{r+1,\Omega}.
\end{aligned}
$$

The term T_{24} is bounded exactly as for T_{22}. Because of the regularity of $\boldsymbol{u}$, the approximation result (2.7), we can bound T_3.

$$
\begin{aligned}
T_3 = \sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{u}_m\cdot\nabla\boldsymbol{\eta}_{m+1})\cdot\boldsymbol{\phi}_{m+1} &+ \sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\boldsymbol{u}_m\}\cdot\boldsymbol{n}_{E_j}|(\boldsymbol{\eta}^{\text{int}}_{m+1}-\boldsymbol{\eta}^{\text{ext}}_{m+1})\cdot\boldsymbol{\phi}^{int}_{m+1} \\
&\leq C\|\boldsymbol{u}_m\|_{L^\infty(\Omega)}h^r|\boldsymbol{u}_{m+1}|_{r+1,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega} \\
&\leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|^2_X + C\nu^{-1}h^{2r}\|\boldsymbol{u}\|^2_{L^\infty([0,T]\times\Omega)}|\boldsymbol{u}_m|^2_{r+1,\Omega}.
\end{aligned}
$$

The term T_4 is bounded using the estimate (2.6).

$$\begin{aligned} T_4 &= \sum_{j=1}^{N_h} \int_{E_j} ((\boldsymbol{u}_{m+1} - \boldsymbol{u}_m) \cdot \nabla \boldsymbol{u}_{m+1}) \cdot \boldsymbol{\phi}_{m+1} \\ &= \int_{t_m}^{t_{m+1}} \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{u}_t \cdot \nabla \boldsymbol{u}_{m+1}) \cdot \boldsymbol{\phi}_{m+1} dt \\ &\leq \Delta t \|\boldsymbol{u}_t\|_{L^\infty(t_m,t_{m+1};L^2(\Omega))} \|\nabla \boldsymbol{u}_{m+1}\|_{L^4(\Omega)} \|\boldsymbol{\phi}_{m+1}\|_{L^4(\Omega)} \\ &\leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\Delta t^2 \|\boldsymbol{u}_t\|^2_{L^\infty(t_m,t_{m+1};L^2(\Omega))} \|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}. \end{aligned}$$

By property of the interpolant (3.11) and properties of $r_h(p)$ (2.9), (2.10), we now bound T_5.

$$\begin{aligned} T_5 &= b(\boldsymbol{\phi}_{m+1}, p^h_{m+1} - (r_h(p))_{m+1}) - b(\boldsymbol{\phi}_{m+1}, p_{m+1} - (r_h(p))_{m+1}) \\ &= -b(\boldsymbol{\phi}_{m+1}, p_{m+1} - (r_h(p))_{m+1}) = \sum_{k=1}^{M_h} \int_{e_k} \{p_{m+1} - (r_h(p))_{m+1}\} [\boldsymbol{\phi}_{m+1}] \cdot \boldsymbol{n}_k \\ &\leq \sum_{k=1}^{M_h} \|[\boldsymbol{\phi}_{m+1}]\|_{0,e_k} |e_k|^{1/2-1/2} \|p_{m+1}\|_{0,e_k} \\ &\leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1} h^{2r} |p_{m+1}|^2_{r,\Omega}. \end{aligned}$$

From a Taylor expansion, we have

$$\begin{aligned} T_6 &\leq C\Delta t \|\boldsymbol{\phi}_{m+1}\|_X \|\boldsymbol{u}_{tt}(t^*)\|_{0,\Omega} \\ &\leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\Delta t^2 \|\boldsymbol{u}_{Tm}\|^2_{L^\infty(0,T;L^2(\Omega))}. \end{aligned}$$

To bound T_7 we assume that $h \leq \Delta t$ and we use the approximation result (2.8) and (2.6).

$$\begin{aligned} T_7 &\leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1} \frac{h^{2r+2}}{\Delta t^2} (|\boldsymbol{u}_{m+1}|^2_{r+1,\Omega} + |\boldsymbol{u}_m|^2_{r+1,\Omega}) \\ &\leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1} h^{2r} (|\boldsymbol{u}_{m+1}|^2_{r+1,\Omega} + |\boldsymbol{u}_m|^2_{r+1,\Omega}). \end{aligned}$$

We expand the term T_8:

$$T_8 = \nu \sum_{j=1}^{N_h} \int_{E_j} \nabla \boldsymbol{\eta}_{m+1} : \nabla \boldsymbol{\phi}_{m+1} - \nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla \boldsymbol{\eta}_{m+1} \boldsymbol{n}_k\} \cdot [\boldsymbol{\phi}_{m+1}]$$
$$+\epsilon_0 \nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla \boldsymbol{\phi}_{m+1} \boldsymbol{n}_k\} \cdot [\boldsymbol{\eta}_{m+1}] + \nu \sum_{k=1}^{M_h} \frac{\sigma}{|e_k|} \int_{e_k} [\boldsymbol{\eta}_{m+1}] \cdot [\boldsymbol{\phi}_{m+1}]$$
$$= T_{81} + \cdots + T_{84}.$$

Clearly, using Cauchy-Schwarz inequality and the approximation result (2.7), we have

$$T_{81} \leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu h^{2r} |\boldsymbol{u}_{m+1}|_{r+1,\Omega}^2.$$

Using the definition of the jump term, the inequality (2.15), and the estimate (2.7), we obtain

$$\begin{aligned} T_{82} &\leq \nu \sum_{k=1}^{M_h} \|[\boldsymbol{\phi}_{m+1}]\|_{0,e_k} (\frac{\sigma}{|e_k|})^{1/2-1/2} \|\nabla \boldsymbol{\eta}_{m+1}\|_{0,e_k} \\ &\leq C\nu J(\boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1})^{1/2} h^r |\boldsymbol{u}_{m+1}|_{r+1,\Omega} \\ &\leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu h^{2r} |\boldsymbol{u}_{m+1}|_{r+1,\Omega}^2. \end{aligned}$$

Using the trace inequalities (2.11), (2.15) and the approximation results (2.7), (2.8), we have

$$T_{83} \leq \nu C \sum_{j=1}^{N_h} \|\boldsymbol{\eta}_{m+1}\|_{0,\partial E_j} \|\nabla \boldsymbol{\phi}_{m+1}\|_{0,\partial E_j}$$
$$\leq \nu C \sum_{j=1}^{N_h} h^{r+1/2} |\boldsymbol{u}_{m+1}|_{r+1,E_j} h^{-1/2} \|\nabla \boldsymbol{\phi}_{m+1}\|_{0,E_j}$$
$$\leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu h^{2r} |\boldsymbol{u}_{m+1}|_{r+1,\Omega}^2.$$

Using the approximation result (2.7), we have

$$\begin{aligned} T_{84} &\leq \nu J(\boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1})^{1/2} J(\boldsymbol{\eta}_{m+1}, \boldsymbol{\eta}_{m+1})^{1/2} \\ &\leq \frac{\kappa\nu}{24} \|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu h^{2r} |\boldsymbol{u}_{m+1}|_{r+1,\Omega}^2. \end{aligned}$$

From the bound (2.7) and the property (4.1), we bound T_9:

$$T_9 \leq \nu_T |||(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|||_0 |||(I-P_H)\nabla\boldsymbol{\eta}_{m+1}|||_0$$
$$\leq \frac{\nu_T}{4}|||(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|||_0^2 + C\nu_T h^{2r}|\boldsymbol{u}_{m+1}|_{r+1,\Omega}^2.$$

Using (4.2), we easily have

$$T_{10} \leq \frac{\nu_T}{4}|||(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|||_0^2 + C\nu_T H^{2r}|\boldsymbol{u}_{m+1}|_{r+1,\Omega}^2.$$

Combining all the bounds of the terms $T_0,\ldots,T_{10}$ yields:

$$\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \frac{\nu\kappa}{2}\|\boldsymbol{\phi}_{m+1}\|_X^2 + \frac{\nu_T}{2}|||(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|||_0^2$$
$$\leq C\nu^{-1}\|\boldsymbol{\phi}_m\|_{0,\Omega}^2 + Ch^{2r}\nu^{-1}(|\boldsymbol{u}_m|_{r+1,\Omega}^2 + |p_{m+1}|_{r+1,\Omega}^2)$$
$$+Ch^{2r}(\nu+\nu^{-1}+\nu_T)|\boldsymbol{u}_{m+1}|_{r+1,\Omega}^2 + C\nu_T H^{2r}|\boldsymbol{u}_{m+1}|_{r+1,\Omega}^2$$
$$+C\nu^{-1}\Delta t^2(\|\boldsymbol{u}_t\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\boldsymbol{u}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}^2).$$

Multiply by $2\Delta t$, sum over m, we obtain:

$$\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_0\|_{0,\Omega}^2 + \nu\kappa\Delta t\sum_{i=0}^{m}\|\boldsymbol{\phi}_{i+1}\|_X^2 + \nu_T\Delta t\sum_{i=0}^{m}|||(I-P_H)\nabla\boldsymbol{\phi}_{i+1}|||_0^2$$
$$\leq Ce^{CT\nu^{-1}}[h^{2r}(\nu+\nu^{-1}+\nu_T)|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2 + \nu_T H^{2r}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2$$
$$+\nu^{-1}\Delta t^2(\|\boldsymbol{u}_t\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\boldsymbol{u}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}^2) + h^{2r}\nu^{-1}|p|_{L^2(0,T;H^r(\Omega))}^2]$$

The final result is obtained by noting that $\|\boldsymbol{\phi}_0\|_{0,\Omega}$ is of order h^r and by using approximation results and a triangle inequality. □

Theorem 6.2. *Assume that* $\boldsymbol{u}_{tt} \in L^\infty(0,T;(H^1(\Omega))^2)$, $p_{tt} \in L^\infty(0,T;H^1(\Omega))$, $\boldsymbol{u}_{ttt} \in L^\infty(0,T;(H^2(\Omega))^2)$ *and* $\boldsymbol{f}_{tt} \in L^\infty(0,T;(L^2(\Omega))^2)$. *Under the assumptions of Theorem 5.1, there is a constant* C *independent of* h, H, ν *and* ν_T *such that*

$$\max_{m=0,\ldots,M}\|\boldsymbol{u}_m - \tilde{\boldsymbol{u}}_m\|_{0,\Omega} + (\nu\kappa\Delta t\sum_{m=0}^{M-1}\|\boldsymbol{u}_{m+1}-\tilde{\boldsymbol{u}}_{m+1}\|_X^2)^{1/2}$$
$$+(\nu_T\Delta t\sum_{m=0}^{M-1}|||(I-P_H)\nabla\boldsymbol{u}_{m+1}-\tilde{\boldsymbol{u}}_{m+1}|||_0^2)^{1/2} \leq Ce^{CT\nu^{-1}}[h^r\nu^{-1/2}\|p\|_{L^2(0,T;H^r(\Omega))}$$
$$+h^r(\nu+\nu^{-1}+\nu_T)^{1/2}\|\boldsymbol{u}\|_{L^2(0,T;H^{r+1}(\Omega))} + \Delta t^2\nu^{1/2}\|\boldsymbol{u}_{ttt}\|_{L^\infty(0,T;H^2(\Omega))}$$
$$+\Delta t^2\nu^{-1/2}(\|\boldsymbol{u}_{tt}\|_{L^\infty(0,T;H^1(\Omega))} + \|p_{tt}\|_{L^\infty(0,T;H^1(\Omega))} + \|\boldsymbol{u}_{ttt}\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{f}_{tt}\|_{L^\infty(0,T;L^2(\Omega))})$$
$$+\nu_T^{1/2}H^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}] + Ch^r|\boldsymbol{u}_0|_{r+1,\Omega}.$$

Proof. The proof is derived in a similar fashion as for the backward Euler scheme. Using the same notation, the error equation is obtained by subtracting the equation (3.6) evaluated at the time $t = t_{m+1/2}$ to the equation (6.3) and adding and subtracting the interpolant $(R_h(\boldsymbol{u}))_{m+1/2}$. After some manipulation, we obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \nu\kappa\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + \nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+\frac{1}{2}}|\!|\!|_0^2 \\
&+c(\tilde{\boldsymbol{u}}^h_{m+\frac{1}{2}}, \boldsymbol{\phi}^h_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}}) \leq |c(\boldsymbol{\phi}_{m+\frac{1}{2}}, \boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})| + |c(\boldsymbol{\phi}_{m+\frac{1}{2}}, \boldsymbol{u}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})| \\
&+|c(\boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{u}^I_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})| + |c(\boldsymbol{u}_{m+\frac{1}{2}}, \boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})| \\
&+|c(\boldsymbol{u}_{m+\frac{1}{2}} - \boldsymbol{u}(t_{m+\frac{1}{2}}), \boldsymbol{u}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})| + |c(\boldsymbol{u}(t_{m+\frac{1}{2}}), \boldsymbol{u}_{m+\frac{1}{2}} - \boldsymbol{u}(t_{m+\frac{1}{2}}), \boldsymbol{\phi}_{m+\frac{1}{2}})| \\
&+|b(\boldsymbol{\phi}_{m+\frac{1}{2}}, \tilde{p}^h_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}}))| + \|\boldsymbol{u}_t(t_{m+\frac{1}{2}}) - \frac{1}{\Delta t}(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m)\|_{0,\Omega}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_{0,\Omega} \\
&+\frac{1}{\Delta t}\|\boldsymbol{\eta}_{m+1} - \boldsymbol{\eta}_m\|_{0,\Omega}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_{0,\Omega} + \|\boldsymbol{f}_{m+\frac{1}{2}} - \boldsymbol{f}(t_{m+\frac{1}{2}})\|_{0,\Omega}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_{0,\Omega} \\
&+\nu|a(\boldsymbol{u}(t_{m+\frac{1}{2}}) - \boldsymbol{u}^I_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+1}) + J(\boldsymbol{u}(t_{m+\frac{1}{2}}) - \boldsymbol{u}^I_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+1})| \\
&+\nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{\eta}_{m+\frac{1}{2}}|\!|\!|_0|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+\frac{1}{2}}|\!|\!|_0 \\
&+\nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{u}_{m+\frac{1}{2}}|\!|\!|_0|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+\frac{1}{2}}|\!|\!|_0 \leq A_0 + \cdots + A_{13}.
\end{aligned}$$

The terms $A_0, A_1, A_2, A_3, A_8, A_{11}$ and A_{12} are bounded exactly like the terms $T_0, T_1, T_2, T_3, T_7, T_9$ and T_{10} respectively. From a Taylor expansion, we bound the terms A_4 and A_5:

$$\begin{aligned}
A_4 + A_5 &= \sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{u}_{m+\frac{1}{2}} - \boldsymbol{u}(t_{m+\frac{1}{2}}))\cdot\nabla\boldsymbol{u}_{m+\frac{1}{2}})\cdot\boldsymbol{\phi}_{m+\frac{1}{2}} \\
&+\sum_{j=1}^{N_h}\int_{E_j}\boldsymbol{u}(t_{m+\frac{1}{2}})\cdot\nabla(\boldsymbol{u}_{m+\frac{1}{2}} - \boldsymbol{u}(t_{m+\frac{1}{2}}))\cdot\boldsymbol{\phi}_{m+\frac{1}{2}} \\
&= \frac{\Delta t^2}{8}\sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{u}_{tt}(t^*))\cdot\nabla\boldsymbol{u}_{m+\frac{1}{2}})\cdot\boldsymbol{\phi}_{m+\frac{1}{2}} + \frac{\Delta t^2}{8}\sum_{j=1}^{N_h}\int_{E_j}\boldsymbol{u}(t_{m+\frac{1}{2}})\cdot\nabla(\boldsymbol{u}_{tt}(t^*))\cdot\boldsymbol{\phi}_{m+\frac{1}{2}} \\
&\leq \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + C\nu^{-1}\Delta t^4\|\boldsymbol{u}_{tt}\|^2_{L^\infty(0,T;H^1(\Omega))}\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}.
\end{aligned}$$

With (3.7), (3.11) and (6.4), the pressure term can be rewritten as:

$$
\begin{aligned}
A_6 &= b(\boldsymbol{\phi}_{m+\frac{1}{2}}, \tilde{p}^h_{m+\frac{1}{2}} - p_{m+\frac{1}{2}}) + b(\boldsymbol{\phi}_{m+\frac{1}{2}}, p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}}))\\
&= -b(\boldsymbol{\phi}_{m+\frac{1}{2}}, p_{m+\frac{1}{2}} - (r_h(p))_{m+\frac{1}{2}}) + b(\boldsymbol{\phi}_{m+\frac{1}{2}}, p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}}))\\
&= \sum_{k=1}^{M_h} \int_{e_k} \{p_{m+\frac{1}{2}} - (r_h(p))_{m+\frac{1}{2}}\}[\boldsymbol{\phi}_{m+\frac{1}{2}}] \cdot \boldsymbol{n}_k - \sum_{j=1}^{N_h} \int_{E_j} (p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}}))\nabla \cdot \boldsymbol{\phi}_{m+\frac{1}{2}}\\
&\quad + \sum_{k=1}^{M_h} \int_{e_k} \{p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}})\}[\boldsymbol{\phi}_{m+\frac{1}{2}}] \cdot \mathbf{n}_k\\
&\leq \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + C\nu^{-1}h^{2r}(|p_{m+1}|^2_{r,\Omega} + |p_m|^2_{r,\Omega}) + C\nu^{-1}\Delta t^4\|p_{tt}\|^2_{L^\infty(0,T;H^1(\Omega))}.
\end{aligned}
$$

We now bound A_7, using a Taylor expansion,

$$
\begin{aligned}
A_7 &\leq C\Delta t^2\|\boldsymbol{u}_{ttt}(t^*)\|_{0,\Omega}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_{0,\Omega}\\
&\leq \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + C\nu^{-1}\Delta t^4\|\boldsymbol{u}_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}
$$

Using also a Taylor expansion, we bound A_9:

$$
A_9 \leq C\nu^{-1}\Delta t^4\|\boldsymbol{f}_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2.
$$

Finally the last term A_{10} is handled as follows:

$$
\begin{aligned}
A_{10} &= \nu[a(\boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}}) + J(\boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})]\\
&\quad + \nu[a(\boldsymbol{u}(t_{m+\frac{1}{2}}) - \boldsymbol{u}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}}) + J(\boldsymbol{u}(t_{m+\frac{1}{2}}) - \boldsymbol{u}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})]\\
&= A_{101} + A_{102}.
\end{aligned}
$$

The term A_{101} is bounded like T_8. The term A_{102} reduces to

$$
\begin{aligned}
A_{102} &= \nu\sum_{j=1}^{N_h}\int_{E_j} \nabla(\boldsymbol{u}(t_{m+\frac{1}{2}}) - \boldsymbol{u}_{m+\frac{1}{2}}) : \nabla\boldsymbol{\phi}_{m+\frac{1}{2}} - \nu\sum_{k=1}^{M_h}\int_{e_k} \{\nabla(\boldsymbol{u}(t_{m+\frac{1}{2}}) - \boldsymbol{u}_{m+\frac{1}{2}})\boldsymbol{n}_k\}[\boldsymbol{\phi}_{m+\frac{1}{2}}]\\
&\leq \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + C\nu\Delta t^4\|\boldsymbol{u}_{tt}\|^2_{L^\infty(0,T;H^2(\Omega))}.
\end{aligned}
$$

Combining all the bounds above yield:

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \frac{\nu\kappa}{2}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + \frac{\nu_T}{2}\|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+\frac{1}{2}}\|\!|_0^2 \\
&\leq C\nu^{-1}(\|\boldsymbol{\phi}_m\|_{0,\Omega}^2 + \|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2) + Ch^{2r}(\nu+\nu^{-1}+\nu_T)(|\boldsymbol{u}_{m+1}|_{r+1,\Omega}^2 + |\boldsymbol{u}_m|_{r+1,\Omega}^2) \\
&\quad + Ch^{2r}\nu^{-1}(|p_{m+1}|_{r,\Omega}^2 + |p_m|_{r,\Omega}^2) + C\Delta t^4\nu\|\boldsymbol{u}_{ttt}\|_{L^\infty(0,T;H^2(\Omega))}^2 \\
&+ C\Delta t^4\nu^{-1}(\|\boldsymbol{u}_{tt}\|_{L^\infty(0,T;H^1(\Omega))}^2 + \|p_{tt}\|_{L^\infty(0,T;H^1(\Omega))}^2 + \|\boldsymbol{u}_{ttt}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\boldsymbol{f}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}^2) \\
&\quad + C\nu_T H^{2r}(|\boldsymbol{u}_{m+1}|_{r+1,\Omega}^2 + |\boldsymbol{u}_m|_{r+1,\Omega}^2).
\end{aligned}
$$

The end of the proof is similar to the one of Theorem 6.1. □

Corollary 6.1. *Assume that $\nu_T = h^\beta$ and $H = h^{1/\alpha}$ where $\beta \geq 2r(\alpha-1)/\alpha$ (see Corollary 5.1), then the estimates in Theorem 6.1 and Theorem 6.2 are optimal.*

$$
\max_{m=0,\dots,M}\|\boldsymbol{u}_m - \boldsymbol{u}_m^h\|_{0,\Omega} + (\Delta t\sum_{m=0}^{M-1}\|\boldsymbol{u}_{m+1} - \boldsymbol{u}_{m+1}^h\|_X^2)^{1/2} = \mathcal{O}(h^r + \Delta t),
$$
$$
\max_{m=0,\dots,M}\|\boldsymbol{u}_m - \tilde{\boldsymbol{u}}_m\|_{0,\Omega} + (\Delta t\sum_{m=0}^{M-1}\|\boldsymbol{u}_{m+1} - \tilde{\boldsymbol{u}}_{m+1}\|_X^2)^{1/2} = \mathcal{O}(h^r + \Delta t^2).
$$

7 Conclusion

In this paper, we have analyzed the stability and convergence of totally discontinuous schemes for solving the time-dependent Navier-Stokes equations. Both semi discrete approximation and fully discrete are constructed for velocity. In addition, semi discrete approximation of pressure is obtained. We showed that these estimations are optimal. Numerical experiments are currently investigated.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] G. Baker, *Galerkin approximations for the Navier-Stokes equations*, Manuscript, Harvard University, 1976.

[3] C. E. Baumann, *An h-p adaptive discontinuous finite element method for computational fluid dynamics*, Ph.D. thesis, The University of Texas, 1997.

[4] G. Birkhoff and G. Rota, *Ordinary differential equations*, Ginn, Boston, 1962.

[5] J. Blasco and R. Codina, *Error estimates for a viscosity-splitting, finite element method for the incompressible Navier-Stokes equations*, submitted to Numerische Mathematik.

[6] J. Boussinesq, *Théorie de l'écoulement tourbillant*, Mem. Pres. par div. savant à l'acad. Sci. Paris (1877).

[7] B. Cockburn, G. Karniadakis, and C. W. Shu, *Discontinuous Galerkin Methods: Theory, Computation, and Applications*, Springer, Berlin, 2000.

[8] M. Crouzeix and R.S. Falk, *Non conforming finite elements for the Stokes problem*, Mathematics of Computation **52** (1989), no. 186, 437–456.

[9] M. Crouzeix and P.A. Raviart, *Conforming and non conforming finite element methods for solving the stationary Stokes equations*, R.A.I.R.O. Numerical Analysis (1973), 33–76.

[10] M. Fortin and M. Soulie, *A non-conforming piecewise quadratic finite element on triangles*, International Journal for Numerical Methods in Engineering **19** (1983), 505–520.

[11] U. Frisch and S. A. Orszag, *Turbulence: challenges for theory and experiment*, Phy. Today (1990), no. 23, 24–32.

[12] V. Girault, B. Rivière, and M. F. Wheeler, *A discontinuous Galerkin method with non-overlapping domain decomposition for the Stokes and Navier-Stokes problems*, to appear in Mathematics of Computation, Technical Report TICAM 02-08, The University of Texas at Austin (2002).

[13] V. Girault and R.L. Scott, *A quasi-local interpolation operator preserving the discrete divergence*, Calcolo **40** (2003), 1–19.

[14] J. L. Guermond, *Stabilization of Galerkin approximations of transport equations by subgrid modelling*, M2AN **33** (1999), 1293–1316.

[15] J. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem part IV: Error analysis for second-order time discretization*, SIAM J. Numer. Anal. **27** (1990), no. 2, 353–384.

[16] T. J. R. Hughes, *The multiscale phenomena: Green's functions, the Dirichlet- to-Neumann formulation, subgrid-scale models, bubbles and the origin of stabilized methods*, Computer Methods in Applied Mechanics and Engineering **127** (1995), 387–401.

[17] T. J. R. Hughes, L. Mazzei, and K. E. Jansen, *Large eddy simulation and the variational multiscale method*, Comput. Visual Sci. **3** (2000), 47–59.

[18] T. J. R. Hughes, L. Mazzei, A. A. Oberai, and A. Wray, *The multiscale formulation of large eddy simulation: decay of homogenous isotropic turbulence*, Physics of Fluids **13** (2001), 505–512.

[19] T. J. R. Hughes, A. A. Oberai, and L. Mazzei, *Large eddy simulation of turbulent channel flows by the variational multiscale method*, Physics of Fluids **13** (2001), 1784–1799.

[20] T. Iliescu and W. J. Layton, *Approximating the larger eddies in fluid motion III: the Boussinesq model for turbulent fluctuations*, Analele Stiintifice ale Universitătii "Al. I. Cuza" Iassi, Tomul XLIV, s.I.a.,Matematica **44** (1998), 245–261.

[21] O. Karakashian and W. Jureidini, *Conforming and non conforming finite element methods for solving the stationary Navier-Stokes equations*, SIAM J. Numerical Analysis **35** (1998), 93–120.

[22] S. Kaya, *Numerical analysis of a subgrid scale eddy viscosity method for higher Reynolds number flow problem*, Technical report, TR-MATH 03-04, University of Pittsburgh, 2003.

[23] S. Kaya and W.J. Layton, *Subgrid-scale eddy viscosity methods are variational multiscale methods*, Technical report, TR-MATH 03-05, University of Pittsburgh, 2003.

[24] W. J. Layton, *A connection between subgrid scale eddy viscosity and mixed methods*, Appl. Math. and Comput. **133** (2002), 147–157.

[25] P. Lesaint and P. A. Raviart, *On a finite element method for solving neutron transport equations, in: Mathematical Aspects of Finite Element Methods in Partial Differential Equations*, C. A. deBoor(Ed) Academic Press (1974), 89–123.

[26] R. Lewandowski, *Analyse Mathématique et Oceanographie*, Masson, Paris, 1997.

[27] Y. Maday and E. Tadmor, *Analysis of spectral vanishing viscosity method for periodic conservation laws*, SIAM J. Numer. Anal. **26** (1989), 854–870.

[28] J. T. Oden, I. Babuska, and C. E. Baumann, *A discontinuous hp finite element method for diffusion problems*, J. Comput. Phys. **146** (1998), 491–519.

[29] W. H. Reed and T. R. Hill, *Triangular mesh methods for the neutron transport equation*, Tech. Report LA-UR-73-479, Los Alamos Scientific Laboratory (1973).

[30] J. Smagorinsky, *General circulation experiments with the primitive equation, I: The basic experiment*, Month. Weath. Rev. **91** (1963), 99–164.

[31] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1995.

[32] M. F. Wheeler, *An elliptic collocation- finite element method with interior penalties*, SIAM J. Numer. Anal. **15** (1978), no. 1, 152–161.