A Mathematical Introduction to Large Eddy Simulation

William J. Layton*†
Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260, U.S.A.

October 16, 2002

Abstract

The purpose of this report is to give a mathematical introduction to large eddy simulation. The treatment of closure focuses on eddy viscosity models and their mathematical foundation.

1 Introduction.

In the numerical solution of turbulent flows, it is usual to attempt to simulate averages of the fluid's velocity rather than its pointwise values. In general, this makes good practical sense: it is widely believed that flow averages evolve according to (hitherto undiscovered) simple deterministic laws while the fluctuations about those averages have a random (but universal) character. When the averages chosen are local, spacial averages, the approach is known as Large Eddy Simulation (LES henceforth). The main claim for LES is that

*wjl@pitt.edu,http://www.math.pitt.edu/∼wjl; partially supported by NSF grants DMS 9972622, INT 9814115 and DMS 0207627

†keywords and phrases: Large eddy simulation, eddy viscosity, Boussinesq approximation, energy inequality

> "LES will simulate the motion of large eddies in a turbulent flow with computational complexity independent of the Reynolds number."

This claim is an exciting challenge to computational scientists to undestand the degree to which it is true and then, with that understanding, to enhance the efficency, reliability and universality of LES.

Herein, we will try to give introduction to LES which will, hopefully, bridge some of the gap between mathematical theory and computational practice. Since turbulence is inherently a multi-disciplinary phenomena, each area can bring interesting insights and tools to its development:

> "We are all led and guided by the passion to perceive and to understand..." L. Euler.

Section 2 reviews aspects of the Navier-Stokes equations that are directly relevant to robust and reliable LES computations. Section 3 presents the basic idea of LES, mainly ignoring boundaries, and Section 4 introduces eddy viscosity models and the Smagorinsky model. Section 5 details possible approaches to improved eddy viscosity models. Conclusions are presented in Section 6.

2 The Navier-Stokes Equations

The history of the development of the Navier-Stokes equations is replete with the names of the great natural philosophers beginning with the first book on mathematical fluid mechanics, *Hydrostatics* by Archimedes. Today, it is widely accepted that the Navier-Stokes equations provide a very accurate description of most flows of almost all liquid and gases. The basic variables are:

$$\begin{aligned}
&\rho : \text{ density}, \quad u = (u_1, u_2, u_3) : \text{ fluid velocity}, \\
&p : \text{ pressure}, \quad \sigma : \text{ stress tensor associated with viscous forces}, \\
&f : \text{ external (body) forces/unit volume.}
\end{aligned}$$

The Navier-Stokes equations are simply a mathematical realization of conservation of mass

$$\rho_t + \nabla \cdot (\rho u) = 0,$$

conservation of linear momentum

$$\rho(u_t + u \cdot \nabla u) - \nabla \cdot \sigma = f,$$

and a linear stress-strain relation

$$\sigma = \mu(\nabla u + \nabla u^{tr}) + (\xi - \frac{2\mu}{3})(\nabla \cdot u)I,$$

where μ and ξ are material parameters known as the first and second viscosities.

The mathematical structure of the NSE is best understood for incompressible fluids. Setting $\rho \equiv \rho_0 =$ constant and nondimensionalizing the resulting equations yields the system we will study herein: the incompressible Navier-Stokes equations

$$\nabla \cdot u = 0 \text{ and } u_t + u \cdot \nabla u + \nabla p - Re^{-1}\Delta u = f, \tag{2.1}$$

where the Reynolds number Re is given by

$$Re = \frac{UL}{\mu/\rho_0} = \frac{\text{characteristic velocity} \times \text{characteristic length}}{\text{kinematic viscosity}}.$$

It is worthwhile for theorists to see a few representative values of Re.

Table 2.1: Representation Values of Re

1 cm. sphere moving 1 cm/sec. in water	$Re \doteq 100$,
subcompact car	$Re \doteq 6 \times 10^5$,
small airplane	$Re \doteq 2 \times 10^7$,
competitive swimmer	$Re \doteq 1 \times 10^6$,
geophysical flows	$Re \doteq 10^{20}$ and higher.

The NSE (2.1) are assumed to hold in the flow domain (hereafter Ω) over some time interval $0 < t \leq T$ and are supplemented by an initial velocity

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2.2}$$

and appropriate boundary conditions. The question of the correct boundary conditions at a wall has been remarkably controversial, [Ser59], [Lia99]. Navier [Nav23] first proposed in 1823 that a fluid at a solid wall must satisfy no-penetration and slip-with-friction conditions, written as

$$u \cdot \hat{n} = 0 \text{ and } \beta u \cdot \hat{\tau} + 2Re^{-1}\hat{n} \cdot \nabla^s u \cdot \hat{\tau} = 0, \text{ and } \partial\Omega, \tag{2.3}$$

where $(\hat{n}, \hat{\tau})$ are the walls unit normal and tangent vectors, β is the friction coefficient and $\nabla^s u = (\nabla u + \nabla u^t)/2$ is the deformation tensor. Stokes proposed both no-penetration and no-slip:

$$u \cdot \hat{n} = 0 \text{ and } u \cdot \hat{\tau} = 0, \text{ on } \partial\Omega. \tag{2.4}$$

The connection (and resolution) was provided by J.C. Maxwell [Max79] in 1879 (!). Deriving the NSE from the kinetic theory of gases by an averaging process, he also recovered Navier's slip law (2.3) where β scaled like

$$\beta \sim \frac{\text{Macroscopic Length Scale}}{\text{Microscopic Length Scale}} \sim Re^{-1} \frac{\text{diam } \Omega}{\text{mean free pass}}.$$

Thus, except for nearly infinite stresses, the no-slip condition (2.4) is the right one.

Of course, a liquid completely enclosed by stationary solid walls is rightly considered an "easy" case. Yet it is still hard enough that analytical and computational studies are done with periodic boundary conditions (to uncouple the equations from the boundaries):

$$\text{(periodic b.c.'s) } u(x + 2\pi, t) = u(x, t) \quad \Omega = (0, 2\pi)^3 \tag{2.5}$$

and (for technical reasons) subject to a zero mean over $(0, 2\pi)^3$ on the solution $u(x, t)$ and on all problem data.

The modern theory of the NSE (indeed of partial differential equations) began with the work of J. Leray [Ler34]. The Leray theory [Gal99] begins with the most concrete and physically meaningful point possible, the global energy inequality, and from that directly constructs the most abstract and (even today) mathematically complete theory of the NSE.

If (u, p) are classical solutions to the NSE subject to either no-slip or periodic boundary conditions, then multiplying (2.1) by u and p respectively, integrating over Ω and applying the divergence theorem immediately shows that

$$\int_\Omega u_t \cdot u + Re^{-1}\nabla u : \nabla u - f \cdot u \, dx = 0.$$

Integrating this in time gives the energy *equality:*

$$k(t) + \int_0^t \epsilon(t')dt' = k(0) + \int_0^t P(t')dt \tag{2.6}$$

where

$$\begin{aligned} k(t) &:= \text{kinetic energy at time } t = \frac{1}{2}\int_\Omega |u|^2 dx, \\ \epsilon(t) &:= \text{energy dissipation rate} := \int_\Omega \epsilon|\nabla u|^2 dx, \\ P(t) &:= \text{power input through force - flow interaction } = \int_\Omega f \cdot u \, dx. \end{aligned}$$

With the energy *equality* (2.1) for classical solutions (which may not exist) Leray was able to prove a much more interesting result.

Theorem 2.1 *(***J. Leray, 1934***). With a given u_0 and f with*

$$\int_\Omega |u|^2 dx < \infty, \int_{\Omega\times(0,T)} |f(x,t)|^2 \, dxdt < \infty.$$

there exists at least one weak solution to the NSE. Weak solutions satisfy the energy inequality

$$k(t) + \int_0^t \epsilon(t')dt \le k(0) + \int_0^t P(t')dt'. \quad \square \tag{2.7}$$

Uniqueness of weak solutions is still not known. (It is a Clay-prize problem with a million dollar reward offered.) Uniqueness appears to be connected to the time regularity of the energy dissipation rate. It is known, for example, that all weak solutions satisfy

$$\int_0^T \epsilon(t')dt' < \infty, \tag{2.8}$$

while weak solutions are unique if, e.g.,

$$\int_0^T \epsilon(t')^2 dt' < \infty. \tag{2.9}$$

Leray in fact conjectured that turbulence was connected to the breakdown of uniqueness in weak solutions to the NSE. In particular, conjecturing that perhaps $\epsilon(t)$ has singularities which are integrable but not square integrable:

(2.8) holds but (2.9) might fail. This conjecture is still an open question and it is still unknown if equality or inequality holds in (2.7), see, e.g., [DR00], Galdi [Gal99] for a very clear elaboration of this theory.

As successful as the Leray theory has been, it has taken many years to begin to establish a connection between it and the Kolmogorov (physical) theory of homogeneous, isotropic turbulence. The status of this connection is well presented in [FMRT01] so we shall skip to the essential elements of Kolmogorov's theory (often called the "K-41" theory) needed in this exposition. For more details see the paper [Kol41] and the very interesting books [Fri95], [Pop00], [Les97]. Consider the NSE under periodic boundary conditions. Let $\mathcal{F}(u) = \hat{u}$ denote the Fourier transform of the velocity field with dual variable $\mathbf{k}$ with $|\mathbf{k}| = (k_1^2 + k_2^2 + k_3^2)^{1/2}$. Define

$$E(k,t) := \frac{1}{2} \int_{|\mathbf{k}|=k} |\hat{u}(\mathbf{k})|^2 \, d\mathbf{k}, \text{ and } E(k) := \lim_{T\to\infty} \frac{1}{T} \int_0^T E(k,t), dt.$$

Data from many different turbulent flows (see, e.g., Figure 7.4 in Frisch [Fri95]) reveal a universal pattern. Plotting the data on $(\log(k), \log\ E(k))$, axes, the universal pattern is a $k^{-5/3}$ decay in $E(k)$ through a wide range of wave-numbers known as the inertial range.

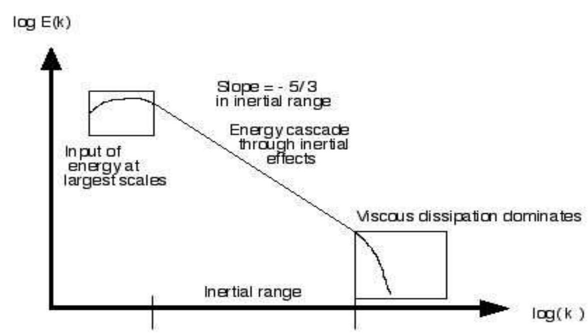


Figure 2.1. A depiction of the observed energy cascade.

By combining Richardson's [Ric65] idea of an energy cascade in turbulent flows with audacious physical guesswork and dimensional analysis, Kolmogorov was able to give a clear explanation of Figure 2.1. Richardson's famous verse on big whirls and lesser whirls was inspired by J. Swift's description of a cascade of poets:

"So, nat'ralists observe a flea

Hath smaller fleas that on him prey;

And these have smaller yet to bite 'em.

And so ad infinitum.

Thus, every poet, in his kind,

Is bit by him that comes behind."

J. Swift

and by L. da Vinci's descriptions of turbulent flows as composed of an area with energy input at the large scales, an area of interactions and an area of decay into small scales:

"where the turbulence of water is generated,

where the turbulence of water maintains for long,

where the turbulence of water comes to rest."

L. da Vinci

Kolmogorov began with the assumption that, (roughly speaking), far enough away from walls, after a long enough time and for high enough Reynolds numbers

time averages of turbulent quantities depend only on one number, the time-averaged energy dissipation rate:

$$\overline{\epsilon} := \lim_{T\to\infty} \frac{1}{T} \int_0^T \epsilon(t)dt. \tag{2.10}$$

Two remarkable consequences were that:

the smallest persistent eddy in a turbulent flow is of diameter $O(Re^{-3/4})$ and $E(k)$ must take the universal form

$$E(k) = \alpha\, \overline{\epsilon}^{2/3}\, k^{-5/3},\ \alpha \cong 1.4, \tag{2.11}$$

with $\overline{\epsilon}$ the only parameter changing from one turbulent flow to another.

The first estimate of $O(Re^{-3/4})$ accounts for the often quoted requirement of $O(Re^{-9/4})$ grid points in space for the direct numerical simulations of a turbulent flow. Considering the magnitudes of representative Reynolds numbers (Table 2.1), it also explains the 1949 assessment of turbulence of von Neumann:

> "It must be admitted that the problems are too vast to be solved by a direct computational attack."
>
> J. von Neumann, 1949,

which is still true today and provides the motivation for the development of LES!

3 The Idea of Large Eddy Simulation

The idea of large eddy simulation is to split the velocity into a local, spacial mean (or average) and a fluctuation about that mean. The mean, $\overline{u}$, is defined by filtering or mollification (convolution with an approximate identity). The goal is to predict the means accurately. This is widely believed possible based on the idea that since the fluctuations have a random character, their *average* effects on the mean notion can successfully be modelled. The first expression of this idea was by an early but famous hydraullic engineer:

> "Observe the notion of the water surface, which resembles that of hair, that has two motions: One due to the weight of the shaft the other to the shape of the curls; thus water has two eddying motions, one part of which is due to the principal current, the other to the random and reverse motion."
>
> L. da Vinci, 1510.

To filter, we must pick a filter. Many different ones are commonly used. To fix ideas we shall use the Gaussian. Choose γ (typically $\gamma = 6$) and define

$$g(x) := \left(\frac{\gamma}{\pi}\right)^{3/2} e^{-\gamma|x|^2} \text{ and } g_\delta(x) := \delta^{-3}\, g\left(\frac{x}{\delta}\right), \quad (\text{if } \Omega \subset \mathbf{R}^3).$$

Then the averaged/filtered velocity is defined by

$$\overline{u}(x,t) := \int_{\mathbf{R}^3} g_\delta(x-y)\, u(y)dy, \text{ and } u' = u - \overline{u},$$

where u is extended off Ω by zero in the case of no-slip boundary conditions and by periodicity under periodic boundary conditions. The mean $\overline{u}(x)$ is a weighted average of u about the point x. As $\delta \to 0$ the points near x are weighted more and more heavily so $\overline{u} \to u$ as $\delta \to 0$. To make this precise it is necessary to introduce some notation (e.g., Galdi [G94], Adams [A75]).

Definition 3.1 *The $L^2(\Omega)$ norm, denoted $||\cdot||$, is*

$$||u|| := \left[\int_\Omega |u|^2 dx\right]^{1/2}.$$

The H^k-norm, denoted $||\cdot||_k$, is

$$||u||_k := \left[\sum_{|\alpha|\le k} ||\frac{\partial^{|\alpha|}}{\partial x^\alpha}\, u||^2\right]^{1/2}.$$

$H^k(\Omega)$ is the closure of the infinitely smooth functions in $||\cdot||_k$.

For constant averaging radius δ a lot is known about filtering, some of which is summarized next.

Theorem 3.1 *Let δ be constant (not varying with position x). Then,*
(a) If $u \in L^2(\Omega)$, $\overline{u} \to u$ as $\delta \to 0$, i.e., $||u-\overline{u}|| \to 0$.
(b) If $u \in L^2(\Omega)$ and $\nabla u \in L^2(\Omega)$ then

$$||u-\overline{u}||_1 \to 0 \text{ as } \delta \to 0.$$

(c) If the velocity field u has bounded kinetic energy then so does $\overline{u}$:

$$\frac{1}{2}\int_\Omega |\overline{u}|^2 dx \le C\frac{1}{2}\int_\Omega |u|^2 dx \tag{3.1}$$

where C is independent of δ.
(d) In the absence of boundaries, (i.e., under periodic boundary conditions) filtering and differentiation commute:

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}\,\overline{u} \;=\; \overline{\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha}\, u\right)}.$$

(e) In the absence of boundaries (i.e., under periodic boundary conditions), for smooth u, $u = \overline{u} + O(\delta^2)$. Specifically,

$$||u-\overline{u}|| \le C\delta^2||u||_2, \text{ for } u \in H^2(\Omega). \tag{3.2}$$

Remarks on the Proof:
Parts (a) - (d) are standard results for averaging by convolution. Part (c) is known at Young's inequality. Part (e) can be proven several different

ways (see e.g., the books [HW55], [Fol95], [Hor90]). For example, using Plancherel's theorem on Fourier transforms:

$$\begin{aligned} ||u-\overline{u}||^2 &= ||\widehat{u-\overline{u}}||^2 = ||(1-\hat{g}_\delta)(\mathbf{k})\hat{u}(\mathbf{k})||^2 = \\ &= \left(\int_{|\mathbf{k}|\geq\pi/\delta} + \int_{|\mathbf{k}|\leq\pi/\delta}\right) |1-\hat{g}_\delta(\mathbf{k})|^2\ |\hat{u}(\mathbf{k})|^2 d\mathbf{k}, \end{aligned} \tag{3.3}$$

where the symbol of the Gaussian is again a Gaussian:

$$\hat{g}_\delta(\mathbf{k}) = e^{-\frac{\delta}{4\gamma}(k_1^2+k_2^2+k_3^2)}$$

On $0 \leq |\mathbf{k}| \leq \pi/\delta$, a Taylor series expansion shows that

$$|1-\hat{g}_\delta(\mathbf{k})|^2 \leq C\delta^4|\mathbf{k}|^4,\ 0 \leq |\mathbf{k}| \leq \pi/\delta, \tag{3.4}$$

while on $|\mathbf{k}| \geq \pi/\delta$

$$|1-\hat{g}_\delta(\mathbf{k})|^2 \leq 2^2 \leq C(1+|\mathbf{k}|)^2)^{-2}(1+|\mathbf{k}|^2)^{+2} \leq C(1+\pi^2\delta^{-2})^{-2}\ (1+|\mathbf{k}|^2)^2.$$

Thus,

$$|1-\hat{g}_\delta(\mathbf{k})|^2 \leq C\ \delta^4(1+|\mathbf{k}|^2)^2,\ \text{for } |\mathbf{k}| \geq \pi/\delta. \tag{3.5}$$

Combining (3.4) and (3.5) in (3.3) gives

$$||u-\overline{u}||^2 \leq C\ \delta^4 \int(1+|\mathbf{k}|^2)^2\ |\hat{u}(\mathbf{k})|^2 d\mathbf{k}.$$

We note that, again by Plancherel's theorem,

$$\int(1+|\mathbf{k}|^2)^2\ |\hat{u}(\mathbf{k})|^2 d\mathbf{k} \leq C\ ||u||_2$$

so that (3.2) follows. □

To calculate $\overline{u}(x,t)$, LES must first construct closed equations for $\overline{u}$. To do this,

(3.6) ignore boundaries and assume constant δ

and convolve the NSE (i.e., $g_\delta * (\text{NSE}) = g_\delta * f$). Using the fact that convolution commutes with differentiation we get the space-filtered-NSE, given by:

$$\begin{aligned} &\overline{u}_t + \nabla\cdot(\overline{u}\ \overline{u}) - Re^{-1}\delta\overline{u} + \nabla\overline{p} + \nabla\cdot\mathcal{R}(u,\ u) = \overline{f}, \\ &\text{and } \nabla\cdot\overline{u} = 0, \end{aligned} \tag{3.7}$$

where $\mathcal{R}(u,\ u)$, the Reynolds stress tensor, is

$$\mathcal{R}(u,\ u) := \overline{\overline{u}\,\overline{u}} - \overline{u}\,\overline{u}. \tag{3.8}$$

Since $\overline{\overline{u}\,\overline{u}} \neq \overline{u}\,\overline{u}$ in general, the usual closure problem has arisen. The first issue in LES is to approximate $\mathcal{R}$ by a tensor depending only on $\overline{u}, \mathcal{S}(\overline{u},\ \overline{u})$, whereupon the closed system:

$$\begin{aligned} &w_t + \nabla \cdot (w\ w) - Re^{-1}\Delta w + \nabla q + \nabla \cdot \mathcal{S}(w,\ w) = \overline{f}, \\ &\text{and } \nabla \cdot w = 0, \end{aligned} \tag{3.9}$$

can be solved. Since $\mathcal{S}(\overline{u},\ \overline{u})$ approximates (but doesn't equal) $\mathcal{R}(u,\ u)$ the solution w of (3.9) is, at best, an approximation to $\overline{u}$ and not $\overline{u}$.

Remark. This description of closure is a slight abuse of the correct continuum mechanical notation. Recall [Ser59] that for incompressible flow, the pressure is the average of the three normal stresses in the coordinate directions. Since $\mathcal{R}$ is a stress, its average contributes to a new pressure. Thus, strictly speaking, the closure problem is to find a tensor, $\mathcal{S}^*(\overline{u},\ \overline{u})$, with zero trace, which approximates $\mathcal{R}^*(u,\ u) := \mathcal{R}(u,\ u) - \frac{1}{3}$ trace $(\mathcal{R}(u,\ u))I$. The new 'pressure' is then $\overline{p} + \frac{1}{3}$ trace $(\mathcal{R}(u,\ u))$.

If we write $u = \overline{u} + u'$ and expand

$$\begin{aligned} &\mathcal{R}(u,\ u) = -\overline{u}\,\overline{u} + \overline{(\overline{u} + u')(\overline{u} + u')} = \\ &-(\overline{u}\,\overline{u} - \overline{\overline{u}\,\overline{u}}) + (\overline{\overline{u}u' + u'\overline{u}}) + \overline{u'\ u'} \end{aligned} \tag{3.10}$$

it is clear that the closure problem can be solved if we can either approximate u in terms of $\overline{u}$ (the de-convolution problem) or approximate u' in terms of its associated $\overline{u}$ (another form of the deconvolution problem). Physically, either reformulation amounts to estimating the effects of the unknowable on the observable. Mathematically, neither can be done in a stable manner in general because of small-divisor problems.

There are many general approaches to closure. We will present three:

- Phenomenological, eddy-viscosity, models
- Asymptotic models
- Models which circumvent closure.

In this report, we shall focus only on the first approach: eddy viscosity models. The second two will be presented in [Lay02a] and [Lay02b].

4 Eddy-Viscosity Models

The first closure problem of LES is thus to find a tensor $\mathcal{S}(\overline{u}\,\overline{u})$ approximating $\mathcal{R}(u,u)$ or at least approximating its effects in the space filtered NSE. To do this, it is useful to have some understanding of the effects of those turbulent fluctuations. Eddy viscosity models are motivated by the following observed experimental behavior (paraphrased form Frisch [F95] who cites it as one of the two experimental laws of turbulence):

Suppose, in an experiment, all control parameters are kept fixed except the viscosity is reduced as far as possible and the energy dissipation is measured (typically by measuring drag). While the flow is laminar, then energy dissipation is reduced proportional to the reduction in ν. When the flow is turbulent, the energy dissipation does not vanish as $\nu \to 0$ but approaches a finite, positive limit.

The action of the Reynolds stresses is thus thought of as having a dissipative effect on the mean flow.

In 1867, Boussinesq [B77] first formulated the eddy-viscosity/Boussinesq hypothesis based upon an analogy between the interaction of small eddies and the perfectly elastic collision of molecules (e.g., molecular viscosity or heat) stating:

Boussinesq Hypothesis (1877) *"Turbulent fluctuations are dissipative in the mean."*

The mathematical realization is the model

$$\nabla \cdot \mathcal{S}(\overline{u},\ \overline{u}) \sim -\nabla \cdot (\nu_T \nabla^s \overline{u}) + \text{ terms incorporated into } \overline{p}, \tag{4.1}$$
$$\nu_T := \text{ Turbulent Viscosity Coefficient } \geq 0.$$

This yields the simple model for $w \cong \overline{u}$.

$$(4.2) w_t + \nabla \cdot (w\ w) - \nabla \cdot ([2Re^{-1} + \nu_T]\nabla^s w) + \nabla q = \overline{f}, \text{ and } \nabla \cdot w = 0.$$

The modeling problem then reduces to determining one parameter: the turbulent viscosity coefficient ν_T:

Closure Problem: Find $\nu_T = \nu_T(\overline{u}, \delta)$.

The global energy balance of eddy viscosity models is very simple and clear.

Proposition 4.1 *Let (w,q) be a classical solution to (3.2) subject to either periodic or no-slip boundary conditions. Let $\nu_T = \nu_T(w,\delta) > 0$. Then,*

$$k(t) + \int_0^t \epsilon_{\mathrm{MODEL}}(t')dt' = k(0) + \int_0^t P(t')dt',$$
where $k(t) = \frac{1}{2}\int_\Omega |w|^2 dx$, $P(t) := \int_\Omega \overline{f} \cdot w \; dx$ and
$$\epsilon_{\mathrm{MODEL}} = \int_\Omega [2Re^{-1} + \nu_T(w, \delta)]\nabla^s w : \nabla^s w \; dx.$$

The most commonly used eddy-viscosity model is known in LES as the Smagorinsky model in which

$$\nu_T = \nu_{\mathrm{Smag}}(w, s) := (C_s\delta)^2|\nabla^s w|.$$

The term $\nabla \cdot ((C_s\delta)^2|\nabla^s w|\nabla^s w)$ was studied in the 1950 by von Neumann and Richtmeyer [vNR50] as a nonlinear artificial viscosity in gas dynamics, by Smagorinsky [Sma63] in 1963 for geophysical flow calculations and a complete mathematical theory for it was constructed around 1964 by Ladzhenskaya (see e.g., [Lad67], [Lad69]), who considered it as a correction term for the linear stress-strain relation for flows with larger stresses. For further mathematical and numerical development of the model we refer to the work of Du and Gunzberger [DG90], [DG91], Pares [Par94] and [Lay96], [JL02].

The modelling difficulty now shifts to determining C_s. The first major result in LES is due to Lilly [Lil67] in 1967 who showed that (under a number of optimistic assumptions) C_s has a simple, universal value 0.17 and is not a "tuning" constant.

Lilly's Estimation of $\mathbf{C_s}$.

The idea of Lilly is to equate $\overline{\epsilon} = \overline{\epsilon}_{\mathrm{MODEL}}$ and from this determine a value for C_s. This approach is very natural: if the model is to give the correct statistics, according to the K-41 theory it must exactly replicate $\overline{\epsilon}$. To explain this idea, we follow closely the presentation in [HMJ00].

Ignoring the viscous dissipation in $\epsilon_{\mathrm{MODEL}}$ (and supressing the time averaging of each term in each step) we can approximate

$$\epsilon_{\mathrm{MODEL}} \cong \int_\Omega \nu_{\mathrm{Smag}}(\delta, w)|\nabla^s w|^2 dx =$$
$$= \int_\Omega (C_s\delta)^2|\nabla^s w|^3 dx = (C_s\delta)^2||\nabla^s w||^3_{L^3}.$$

If we assume that the time average of w is exactly the same as that of u restricted to the frequencies $0 \le |\mathbf{k}| \le k_c := \pi/\delta$ then Plancherel's theorem gives (in the time averaging sense)

$$\begin{aligned} ||\nabla^s w||^2 &\doteq \int_0^{k_c} k^2 E(k) dk = \int_0^{k_2} k^2 (\alpha\epsilon^{2/3} k^{-5/3}) dk \\ &= \frac{3}{4}\,\alpha\epsilon^{2/3} k_c^{4/3},\ k_c = \pi/\delta. \end{aligned}$$

If we assume that for homogeneous, isotropic turbulence, after time averaging,

$$||\nabla^s w||_{L^3}^3 \cong ||\nabla^s w||^3.$$

then we can write

$$\epsilon_{\text{MODEL}} \cong (C_s\delta)\left[\frac{3}{4}\,\alpha\epsilon^{2/3}k_c^{4/3}\right]^{3/2},\ \text{where } k_c = \pi/\delta,\ \text{or}$$

$$\epsilon_{\text{MODEL}} \cong C_s^2 \pi^2 \left(\frac{3}{4}\right)^{3/2} \alpha^{3/2}\epsilon,\ \alpha \doteq 1.4.$$

Equating $\epsilon = \epsilon_{\text{MODEL}}$, all dependence on ϵ cancels in the equation giving

$$C_s = \pi^{-1}\left(\frac{4}{3}\right)^{3/4} \alpha^{-3/4} \cong 0.17.$$

The universal value 0.17, independent of the particular flow, is obtained. This is often expressed as

"Smagorinsky is consonant with Kolmogorov."

Interestingly, this universal value $C_s = 0.17$ has almost universally (in experiment) been found to be too large. There have been many other criticisms of the Smagorinsky model associated with it being too dissipative. Rather than summarize them here, we present below two simulations of a $2d$ flow over an obstacle: one a DNS and the other with the Smagorinsky model. It is clear from these that the dissipartion in this model is too powerful.

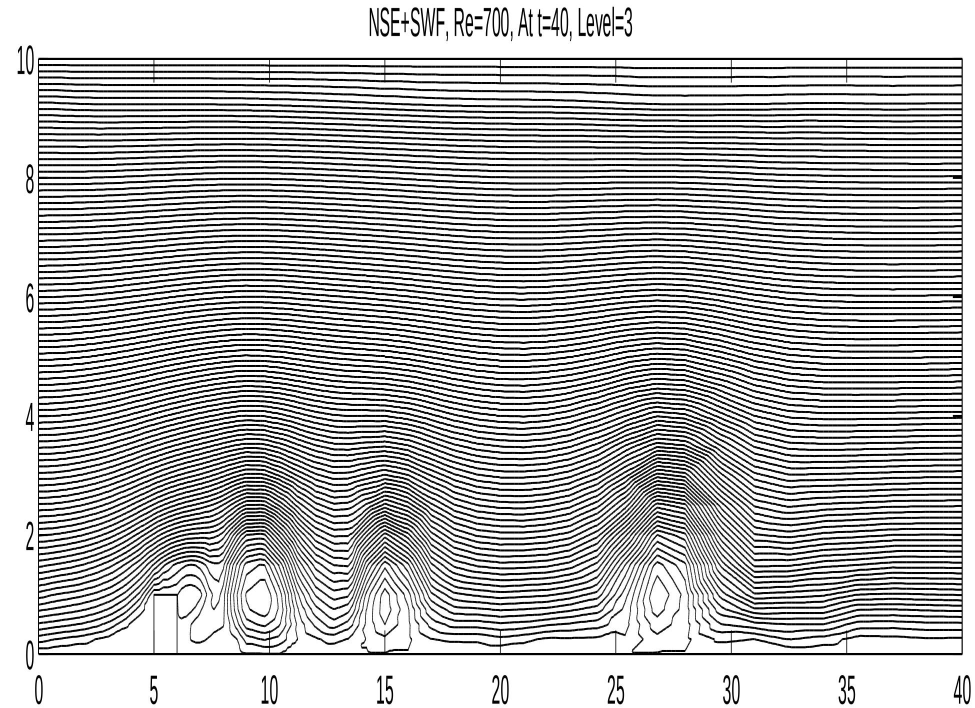


Figure 4.1.a. Streamlines of the true solution at time t=40 (N. Sahin, 2002)

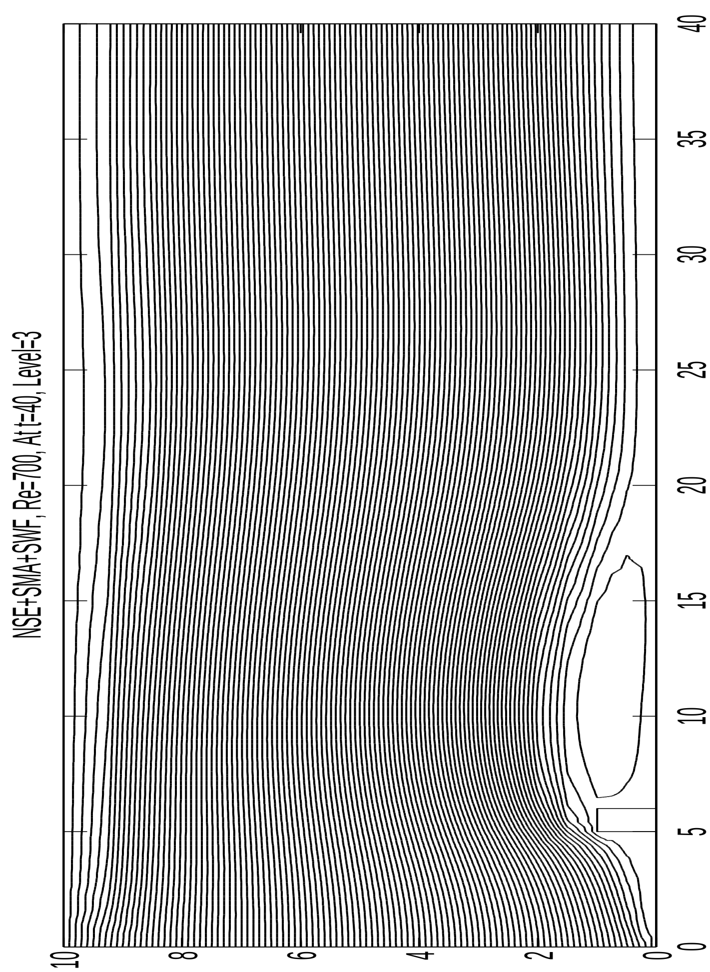


Figure 4.1.b. Streamlines of the Smagorinsky Model at time t=40 (N. Sahin, 2002).

Germano's idea of dynamic parameter selection, e.g., [GPMC91], gives a big improvement in the performance of the Smagorinsky model. We will not delve into dynamic models here for two reasons. First, dynamic parameter selection is really a way to *improve* the performance of almost any model (and is not specific to the Smagorinsky model). Second, its mathematical foundation seems to be beyond the tools presently available.

5 Improved Eddy Viscosity Models

The connection between turbulent fluctuations and the choice

$$\nu_T = (C_s\delta)^2|\nabla^s\overline{u}|$$

of the Smagorinsky's model eddy viscosity seems tenuous. Thus, it is natural to seek other choices of ν_T with a more direct connection. Boussinesq based his model upon the analogy between perfectly elastic collisions and interaction of small eddies.

Within this reasoning (whose "optimism" he acknowledged) it is clear that the amount of turbulent mixing should depend mainly on the local kinetic energy in the turbulent fluctuations:

$$\nu_T = \nu_T(\delta, \overline{k'}),\ k'(x) = \frac{1}{2}|u'|^2(x). \tag{5.1}$$

The simplest functional form which is dimensionally consistent is the, so-called, Kolmogorov-Prandtl relation, given by

$$\nu_T \doteq C\delta\sqrt{\overline{k'}},\ k' = \frac{1}{2}|u'|^2. \tag{5.2}$$

One method of estimating k' is to solve an approximate energy equation. The recent important work of Duchon and Robert [DR00] shows the correctness of this approach to be unclear. However, the most simple and direct estimate of k' is obtained by scale similarity: the best estimate for the kinetic energy in the unresolved scales is that of the smallest resolved scales, [IL98]. Mathematically,

$$\overline{k'} = \overline{\frac{1}{2}|u-\overline{u}|}^2 \ \sim\ \frac{1}{2}|\overline{u}-\overline{\overline{u}}|^2.$$

Inserting this estimate into (5.2) gives the LES-eddy viscosity model:

$$\begin{aligned} &w_t + \nabla\cdot(w\ w) + \nabla q - Re^{-1}\Delta w - \nabla\cdot(\mu_0\delta|w-\overline{w}|\nabla^s w) = \overline{f}, \\ &\text{and } \nabla\cdot w = 0. \end{aligned} \tag{5.3}$$

The parameter μ_0 can either be determined dynamically or estimated adapting the approach of Lilly.

This eddy viscosity model is much less dissipative than the Smagorinsky model. Indeed,

$$\nu_{\text{Smag}} = (C_s\delta)^2|\nabla^s w| = \begin{cases} O(\delta^2) & \text{in smooth regions,} \\ O(\delta) & \text{for fluctuations,} \end{cases}$$

while, (recall that $||w-\overline{w}|| = O(\delta^2)$ in smooth regions)

$$\nu_T = \mu_0 \delta |w - \overline{w}| = \begin{cases} O(\delta^3) & \text{in smooth regions,} \\ O(\delta) & \text{for fluctuations.} \end{cases} \tag{5.4}$$

Since (5.3) is an eddy viscosity model, its energy budget is clear (Proposition 4.1). Nevertheless, the fact that $\nu_T(w)$ can be unbounded places the model (5.3) outside the usual Leray-Lions theory for verifying existence of a distributional solution to the model. The mathematical elucidation of this model (5.3) was begun in [LL02]. It is again based upon the global energy equality of eddy viscosity methods.

Definition 5.1 *Let $\mu_0 > 0$ be fixed and consider (5.3) subject to periodic boundary conditions. Then, w is an distributional solution of (5.3) if*

$$w \in Y := \textit{ closure } \{v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) : \nabla \cdot w = 0\}$$
$$\textit{and if for all } \phi \in \{\psi \in C^\infty(0,T;C^\infty_{periodic}(\Omega)) : \nabla\cdot\psi = 0 \textit{ and } \psi(T,\cdot) = 0\}$$
$$\int_\Omega \overline{u}_0(x)\cdot\phi(x,0)dx - \int_0^T \int_\Omega w \, \frac{\partial \phi}{\partial t} \, dxdt - \int_0^T \int_\Omega ww : \nabla\phi$$
$$+\nu_T(w)\nabla^s w : \nabla^s \phi \; dxdt = \int_0^T \int_\Omega \overline{f}\cdot\phi \; dxdt.$$

In [LL01] existence of distributional solutions to the model (5.3) was proven.

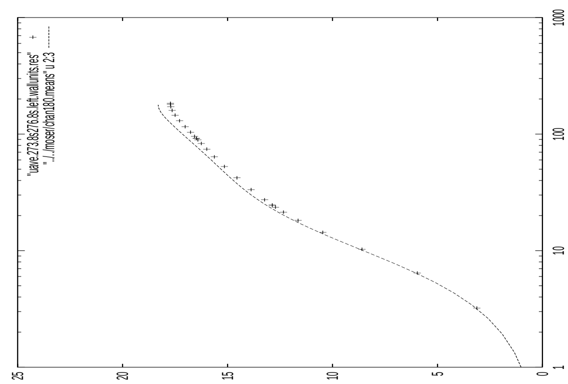
Theorem 5.1 (Theorem 3.1 of [LL01].) *For $\overline{u}_0 \in L^2(\Omega)$, $\overline{f} \in L^2(\Omega \times (0,T))$ there exists at least one distributional solution to the model (5.3).* □

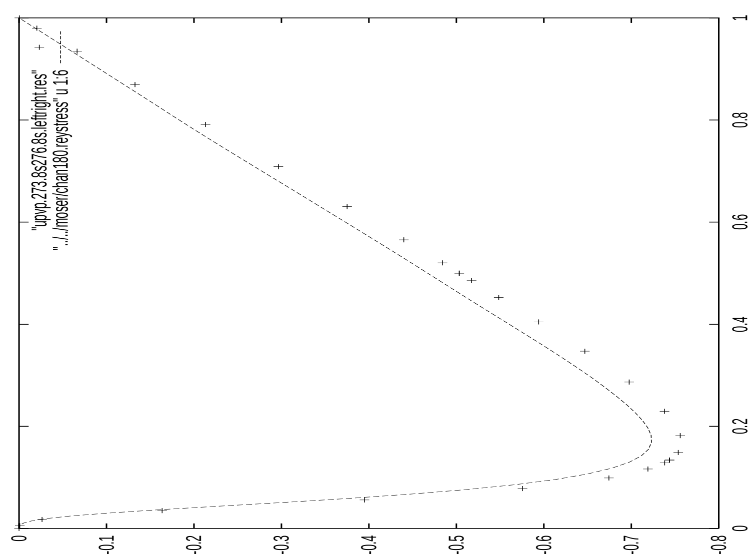
The theory behind this result also includes many filters, even differential filters, and many eddy viscosities $\nu_T(w)$ which minimally satisfy the following three consistency and growth conditions: for all $w \in Y$

- $\nu + \nu_T(w) \geq C_0 > 0$,
- $\nu_T(w)(x,t) \in L^\infty(0,T;L^2(\Omega))$, and
- $||\nu_T(w)||_{L^\infty(0,T;L^2(\Omega))} \leq C(1 + ||w||_{L^\infty(0,T;L^2(\Omega))})$.

Experiments with the model have, so far, been positive. Tests of turbulent channel flow of T. Iliescu and P. Fischer indicate that it replicates the standard turbulent statistics well.

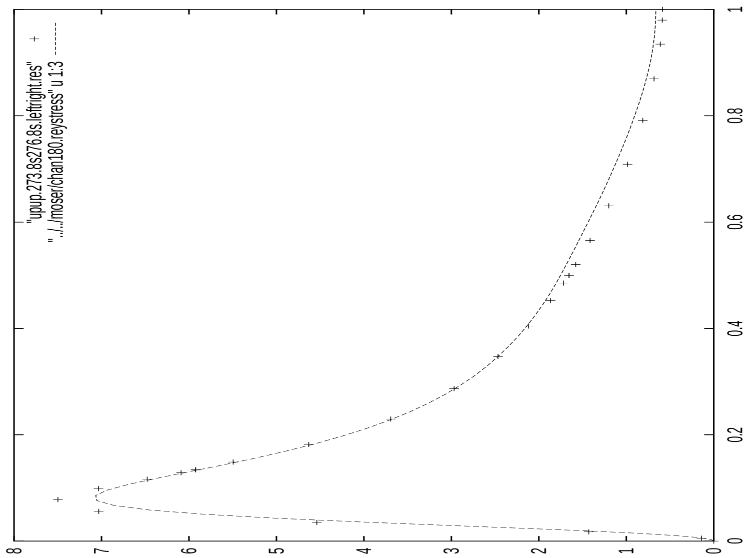
The next three figures, provided courtesy of T. Iliescu and P. Fischer, show respectively, the models predicted mean velocity vs. true (from a filtered DNS database), the model's predicted values of $u'v'$ vs. the true values





and, third, the model's predicted values of $u'u'$ vs. the true values, again from the filtered DNS database of Moser, Kim and Mansour [MKM99].
V. John [J02] has also conducted extensive test of $\nu_T = \mu_0\delta|w - \overline{w}|$ as an eddy viscosity term incorporated into a mixed model, also with good results.

The eddy viscosity $\nu_T = \mu_0\delta|w - \overline{w}|$ has the simplest form and most direct connection with the physical ideas of turbulent mixing, so it is not surprising that closely related models have been independently tested in practical computations. In particular, interesting work has been done by Horuiti [Hor85] upon scale-similarity models in general and models like the present ν_T in particular and Sagant [Sag96] who has tested geometric averages of ν_T and



ν_{Smag}:

$$\nu = \nu_T^{\theta}\ \nu_{\text{Smag}}^{(1-\theta)} = C\delta^{2-\theta}|w - \overline{w}|^{\theta}|\nabla^s w|^{1-\theta},$$

in some very challenging compressible flow problems.

Dimensionally Equivalent Models

Not all models that are *dimensionally equivalent* can be expected to *perform analogously.* Thus, there is a real interest in exploring dimensionally equivalent versions of the model to test their differences, relative advantages and disadvantages. The three which come immediately to mind are:

(5.5) $\nu_T = \mu \cdot \delta |w - \overline{w}|$, the model (5.3),

(5.6) $\nu_T = \mu_1 \delta^2 |\nabla^s (w - \overline{w})|$, i.e., $\nabla \cdot (\mu_1 \delta^2 |\nabla^s (w - \overline{w})| \nabla^s w)$,

(5.7) $\nu_T = \mu_2 \delta^3 |\Delta (w - \overline{w})|$.

The model (5.6) was studied and tested by Hughes, Mazzei and Jansen [HMJ00] who called it the "small-large Smagorinsky model". The model (5.7) seems appealing computationally, but it seems that a mathematical development of it is very difficult. The Gaussian-Laplacian model of [IL98], which we present next, is a better candidate for a robust model.

A Related Model: The Gaussian-Laplacian Model

The Gaussian-Laplacian model is based on an asymptotic expansion of $\nu_T = \mu_0 \delta |w - \overline{w}|$ in wave number space as follows. Since $u' = u - \overline{u}$, taking Fourier transforms we have

$$\begin{aligned} \mathcal{F}(u') &= \mathcal{F}(u - \overline{u}) = \hat{u} - \hat{g}_\delta(\mathbf{k})\hat{u} = \\ &= (\hat{g}_\delta^{-1}(\mathbf{k}) - 1)\hat{g}_\delta(\mathbf{k})\hat{u}, \text{ or,} \\ \hat{u'} &= (\hat{g}_\delta(\mathbf{k})^{-1} - 1)\hat{\overline{u}}(\mathbf{k}). \end{aligned} \tag{5.8}$$

The *key feature* of the Gaussian is its smoothing property which is equivalent to the decay of $\hat{g}_\delta(\mathbf{k})$ as $|\mathbf{k}| \to \infty$. Thus, Taylor approximations (which have the opposite behavior) are not appropriate. The simplest approximation preserving this is the subdiagonal $(0,1)$-Padé approximation, e.g., [GL00], [Poz94],

$$\hat{g}_\delta(\mathbf{k}) \cong \frac{1}{1 + \frac{\delta^2}{4\gamma}|\mathbf{k}|^2} + O(\delta^4). \tag{5.9}$$

The approximation (5.9), used in (5.8), gives:

$$\hat{u'}(\mathbf{k}) = \frac{\delta^2}{4\gamma} |\mathbf{k}|^2 \hat{\overline{u}}(\mathbf{k}) + O(\delta^4).$$

This gives the approximation to the kinetic energy of the turbulent fluctuations:

$$\overline{k'} = \frac{1}{2} \frac{\delta^2}{4\gamma} |\overline{\Delta w}|^2 + O(\delta^4).$$

and the turbulent viscosity ([IL98])

$$\nu_T = \mu_3 \frac{\delta^3}{4\gamma} |g_\delta * \Delta w|. \tag{5.10}$$

The resulting model is:

$$\begin{gathered} w_t + \nabla \cdot (w\ w) - \nabla q - Re^{-1}\Delta w - \nabla \cdot (\mu_3 \frac{\delta^3}{4\gamma} |g_\delta * \Delta w| \nabla^s w) = \overline{f}, \\ \text{and } \nabla \cdot w = 0. \end{gathered} \tag{5.11}$$

The eddy viscosity (5.10) is bounded, thanks to the regularization via convolution by a gaussian. Thus, it was possible in [IL98] to extend the Leray-Lions theory of weak solutions of the NSE to the model (5.11).

Theorem 5.2 *Consider (5.11) subject to periodic boundary conditions and* $\mu_3 > 0, \gamma > 0$. *Then, a weak solution to (5.11) exists in the large and satisfies the energy inequality*

$$k(t) + \int_0^t \epsilon_{\text{Model}}(t')dt' \leq k(0) + \int_0^t P(t')\, dt',$$

where $k(t) := \frac{1}{2}\int_\Omega |w|^2 dx,\ P(t) := \int_\Omega \overline{f} \cdot w\, dx$ and

$$\epsilon_{\text{Model}}(t) := \int_\Omega Re^{-1}|\nabla w|^2 + \mu_3 \frac{\delta^3}{4\gamma}|g_\delta * \Delta w|\ |\nabla^s w|^2\, dx. \quad \square$$

The extra eddy viscosity term in (5.11) is called a Gaussian-Laplacian. It has other interesting mathematical properties and has been used for image-smoothing by, e.g., [CLMC92], [MS98].

Remark: Deconvolution by Padé Methods.

The problem of closure is solved provided a "good" deconvolution approximation $u \doteq O(\overline{u})$ can be obtained. This point of view has been extensively developed by Stoltz and Adams [SA97]. The considerations leading to (5.9) (and developed in [GL00]) suggests that deconvolution of Gaussian filters using Padé approximations rather than Taylor approximations is the correct approach. For example, we have

$$\hat{\overline{u}} = \hat{g}_\delta\ \hat{u} \doteq \left(\frac{1}{1 + \frac{\delta^2}{4\gamma}|\mathbf{k}|^2}\right) \hat{u}\ (+O(\delta^4))$$

or $\hat{u} \doteq (1 + \frac{\delta^2}{4\gamma}\ |\mathbf{k}|^2)\hat{\overline{u}}$, which, when inverted, gives

$$u \doteq (-\frac{\delta^2}{4\gamma}\ \Delta + 1)\ \overline{u}.$$

6 Conclusions

The fundamental problems of LES revolve around *closure* and there are several closure problems, including

- Subgrid scale models for the Reynolds stresses

$$\mathcal{R}(u,u) \Leftarrow\ \mathcal{S}(\overline{u},\ \overline{u}).$$

• Closure for the extra terms arising because convolution/filtering and differentiation *do not commute* on bounded domains ([DJL02]).

• Closure for extra terms which arise with variable averaging radius $\delta = \delta(x)$, due to the same non-commutativity, (see the interesting work of Vasilyev, Lund and Moin [VLM98]).

• The problem of Near Wall Modeling and finding *local* boundary conditions for *non-local* averages of fluid velocities.

Eddy-viscosity models have been a workhorse for the first closure problem in LES. Even with them, there remains a lot to be done to increase their robustness, accuracy and universality. The most interesting new developments are models whose turbulent viscosity coefficients depend only on the finest resolved scales (discussed herein) and models acting only on the finest resolved scales (discussed in a subsequent lecture). However, the ultimate usefulness of eddy viscosity models will likely be as a model of the turbulent fluctuation interaction term $-\nabla \cdot (\overline{u' \, u'})$ in (3.10):

$$-\nabla \cdot (\overline{u' \, u'}) \sim -\nabla \cdot (\nu_T \nabla^s \, \overline{u}),$$

which is combined with more systematic models of the remaining terms in (3.10) to give a mixed model.

In computations and in the mathematical development of the models, understanding the model's global energy inequality is the key to insight and progress.

Acknowledgements

My understanding continues to grow thanks to interactions with many colleagues. In particular, I am happy to thank my collaborators on the topics of these lecture notes, G.P. Galdi, L. Tobiska, R. Lewandowski, L.C. Berselli, T. Iliescu, V. John, N. Sahin, A. Dunca and S. Kaya.

References

[Ada75] R. A. Adams. *Sobolev spaces.* Academic Press, New York, 1975.

[Ald90] A. A. Aldama. Filtering techniques for turbulent flow simulation. In *Springer Lecture Notes in Eng.*, volume 56. Springer, Berlin, 1990.

[BBG02] D. Barbato, L.C. Berselli, and C.R. Grisanti. Analytical and numerical results regarding the spectral approximation of the rational LES model. in preparation, 2002.

[BC98] G.I. Barenblatt and A.J. Chorin. New Perspectives in Turbulence Scaling laws, asymptotics, and intermittency. *SIAM Review*, 40(2):265–291, 1998.

[BFR80] J. Bardina, J.H. Ferziger, and W.C. Reynolds. Improved subgrid scale models for large eddy simulation, 1980.

[BG02] L. C. Berselli and C. Grisanti. On the consistency of the Rational large eddy simulation model. Technical report, University of Pisa, 2002.

[BGIL02] L.C. Berselli, G.P. Galdi, T. Iliescu, and W.J. Layton. Mathematical analysis for the Rational Large Eddy Simulation Model. *Math. Models Methods Appl. Sci.*, 12:1–22, 2002.

[BI02] L.C. Berselli and I. Iliescu. A higher order subfilter-scale model for large eddy simulation. tech. report, 2002.

[Bou77] J. Boussinesq. Essai sur la théorie des eaux courantes, meins. prés par div. sarants á la. *Acad. Sci.*, 23:1–680, 1877.

[CLMC92] F. Catté, P.L. Lions, J. M. Morel, and T. Coll. Image selective smoothing and edge detection by nonlinear diffusion. *SIAM J.N.A.*, 129:182–193, 1992.

[CMR98] T.A. Clopeau, T.A. Mikelic, and R. Robert. On the vanishing viscosity limit for the 2d incompressible Navier-Stokes equations with the friction type boundary conditions. *Nonlinearity*, 11:1625–1636, 1998.

[Col98] P. Coletti. *Analytical and Numerical Results for k-ϵ and large eddy simulation turbulence models*. PhD thesis, Universitá di Trento, 1998.

[DG90] Q. Du and M. D. Gunzburger. Finite-element approximations of a Ladyzhenskaya model for stationary incompressible viscous flow. *SIAM J. Numer. Anal.*, 27:1–19, 1990.

[DG91] Q. Du and M. D. Gunzburger. Analysis of a Ladyzhenskaya model for incompressible viscous flow. *J. Math. Anal. Appl.*, 155:21–45, 1991.

[DJL02] A. Dunca, V. John, and W. J. Layton. The commutation error of the space averaged Navier-Stokes equations on a bounded domain, 2002. Preprint, University of Pittsburgh.

[DR00] J. Duchon and R. Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity*, 13:249–255, 2000.

[FHT01] C. Foias, D.D. Holm, and E.S. Titi. The Navier-Stokes alpha model of fluid turbulence. *Phys. D.*, 152/153:505–519, 2001.

[FMRT01] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier Stokes Equations and Turbulence.* Cambridge University Press, 2001.

[Fol95] G. B. Folland. Introduction to Partial Differential Equations. In *Mathematical Notes*, volume 17. Princeton University Press, 2nd edition, 1995.

[Fri95] U. Frisch. *Turbulence, The Legacy of A. N. Kolmogorov.* Cambridge, 1995.

[Gal94] G. P. Galdi. An Introduction to the Mathematical Theory of the Navier-Stokes Equations, vol i. In *Linearized Steady Problems*, volume I. Springer Tracts in Natural Philosophy, 1994.

[Gal99] G. P. Galdi. *Lectures in Mathematical Fluid Dynamics.* Birkhauser-Verlag, 1999.

[Ger86a] M. Germano. Differential filters for the large eddy simulation of turbulent flows. *Phys. Fluids*, 29(6):1755–1757, 1986.

[Ger86b] M. Germano. Differential filters of elliptic type. *Phys. Fluids*, 29(6):1757–1758, 1986.

[GL00] G.P. Galdi and W.J. Layton. Approximating the larger eddies in fluid motion II: A model for space filtered flow. M^3AS, 10:1–8, 2000.

[GPMC91] M. Germano, U. Piomelli, P. Moin, and W. Cabot. A dynamic subgrid-scale eddy viscosity model. *Phys. Fluids A*, 3:1760–1765, 1991.

[HMHMR98] T.J.R. Hughes, L. Mazzei, D.D. Holm, J.E. Marsden, and T.S. Ratiu. Euler-Poincaré models of idea fluids with nonlinear dispersion. *Phys. Rel. Lett.*, 80:4173–4176, 1998.

[HMJ00] T. J. Hughes, L. Mazzei, and K. E. Jansen. Large eddy simulation and the variational multiscale method. *Comput. Visual Sci.*, 3:47–59, 2000.

[HMOW01] T. J. Hughes, L. Mazzei, A. Oberai, and A. Wray. The multiscale formulation of large eddy simulation: Decay of homogeneous isotropic turbulence. *Phys. Fluids*, 13(2):505–512, 2001.

[HOM01] T. J. Hughes, A. Oberai, and L. Mazzei. Large eddy simulation of turbulent channel flows by the variational multiscale method. *Phys. Fluids*, 13(6):1784–1799, 2001.

[Hor85] K. Horuiti. Large eddy simulation of turbulent channel flow by one equation modeling. *J. Phys. Soc. Japan*, 54:2855–2865, 1985.

[Hor90] L. Hörmander. *The Analysis of Partial Operators I.* Springer-Verlag, Berlin, 2nd edition, 1990.

[HW55] I.I. Hirschman and D. V. Widder. *The Convolution Transform*, 1955.

[IJL02] T. Iliescu, V. John, and W.J. Layton. Convergence of Finite Element Approximations of Large Eddy Motion. *Num. Meth. Part. Diff. Eqn., accepted for publication*, 2002.

[IL98] T. Iliescu and W. J. Layton. Approximating the larger eddies in fluid motion iii: the Boussinesq model for turbulent fluctuations. *Analele Stiintifice ale Universităţii "Al. I. Cuza" Iasşi, Tomul XLIV, s.I.a., Matematică*, 44:245–261, 1998.

[Ili00] T. Iliescu. Large Eddy Simulation for Turbulent Flows, 2000. Ph. D. thesis, Univ. of Pittsburgh.

[Jim99] J. Jimenez. An overview of LES validation. In *A Selection of Test Cases for the Validation of Large Eddy Simulation of Turbulent Flows*. NATO AGARD, 1999.

[JL02] V. John and W. J. Layton. Analysis of numerical errors in large eddy simulation. *to appear in: SIAM J. Numer. Anal.*, 2002.

[JLS02] V. John, W. J. Layton, and N. Sahin. Derivation and analysis of near wall models for channel and recirculating flows. Technical report, University of Pittsburgh, 2002.

[Joh02a] V. John. The behaviour of the rational LES model in a two-dimensional mixing layer problem, 2002. tech. report.

[Joh02b] V. John. Large Eddy Simulation of Turbulent Flows, Analytical and Numerical Results for a class of LES Models, 2002. Habilitation thesis: Otto-von-Guericke Univ., Magdeburg, Germany.

[Joh02c] V. John. Slip with friction and penetration with resistance boundary conditions for the Navier-Stokes equations - numerical tests and aspects of the implementation. *to appear in: J. Comp. Appl. Math.*, 2002.

[Kol41] A. V. Kolmogorov. The local structure of turbulence in incompressible viscous fluids for very large Reynolds number. *Dokl. Akad. Nauk SSR*, 30:9–13, 1941.

[Lad67] O. A. Ladyzhenskaya. New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them. *Proc. Steklov Inst. Math.*, 102:95–118, 1967.

[Lad69] O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, 2nd edition, 1969.

[Lay96] W. J. Layton. On nonlinear subgrid scale models for viscous flow problems. *SIAM J. Sci. Computing*, 17:347–357, 1996.

[Lay99] W.J. Layton. Weak imposition of No-Slip conditions in finite element methods. *Computers and Mathematics with Applications*, 38:129–142, 1999.

[Lay00] W. Layton. Approximating the larger eddies in fluid motion V: kinetic energy balance of scale similarity models. *Mathematical and Computer Modelling*, 31:1–7, 2000.

[Lay02a] W. Layton. Advanced models in large eddy simulation. *VKI Lecture Notes*, 2002.

[Lay02b] W. Layton. Variational multiscale methods and subgrid scale eddy viscosity. *VKI Lecture Notes*, 2002.

[Ler34] J. Leray. Sun le mouvement d'un fluide visqueux emplissant l'espace. *Acta Math.*, 63:193–248, 1934.

[Les97] M. Lesieur. Turbulence in Fluids. In *Fluid Mechanics and its Applications*, volume 40. Kluwer Academic Publishers, 1997.

[Lia99] A. Liakos. *Weak imposition of boundary conditions in the Stokes and Navier-Stokes equation.* PhD thesis, University of Pittsburgh, 1999.

[Lil67] D. K. Lilly. The representation of small-scale turbulence in numerical simulation experiments. *Proc. IBM Sci. Comp. Symp. on Eviron. Sci.*, 1967.

[LL01] W.J. Layton and R. Lewandowski. Analysis of an eddy viscosity model for large eddy simulation of turbulent flows. *to appear in: J. Math. Fluid Mechanics*, 2001.

[Max79] J.C. Maxwell, 1879. Phil. Trans. Royal Society.

[MKM99] D. R. Moser, J. Kim, and N. N. Mansour. Direct numerical simulation of turbulent channel flow up to re-tan $= 590$. *Phys. Fluids*, 11:943, 1999.

[MP94] B. Mohammadi and O. Pironneau. *Analysis of the K-Epsilon Turbulence Model.* John Wiley and Sons, 1994.

[MS98] K. Mikula and F. Sgallari. Computational Methods for nonlinear diffusion equations on image processing, 1998. tech. report.

[Nav23] C.L.M.H. Navier. Mémoire sur les lois du mouvement des fluiales. *Mém. Acad. Royal Society*, 6:389–440, 1923.

[Par94] C. Pares. Approximation De La Solution Des Equations D'un Modele De Turbulence par Une Methode De Lagrange-Galerkin. *Rev. Mat. Apl.*, 15:63–124, 1994.

[PB02] U. Piomelli and E. Balaras. Wall-layer models for Large-eddy simulations. *Annu. Rev. Fluid Mech.*, 34:349–374, 2002.

[Pop00] S. B. Pope. *Turbulent flows*. Cambridge University Press, 2000.

[Poz94] A. Pozzi. *Applications of Padé approximation theory in fluid dynamics*. World Scientific, 1994.

[Ric65] L. F. Richardson. *Weather Prediction by Numerical Processes*. Dover, New York, 1965.

[SA99] S. Stoltz and N. A. Adams. An approximate deconvolution procedure for large-eddy simulation. *Phys. Fluids*, 11:1699–1701, 1999.

[Sag96] P. Sagaut. Numerical studies of separated flows with subgrid models. *Rech. Aéro*, 1:51–63, 1996.

[Sag01] P. Sagaut. *Large Eddy Simulation for Incompressible Flows*. Springer, Berlin, 2001.

[Ser59] J. Serrin. *Mathematical Principles of Classical Fluid Mechanics*, pages 125–263. Springer, 1959.

[Sma63] J. S. Smagorinsky. General circulation experiments with the primitive equations. *Mon. Weather Review*, 91:99–164, 1963.

[VLM98] O. V. Vasilyev, T. S. Lund, and P. Moin. A general class of commutative filters for LES in complex geometries. *Journal of Computational Physics*, 146:82–104, 1998.

[vNR50] J. von Neumann and R. D. Richtmyer. A method for the numerical calculation of hydrodynamic shocks. *J. Appl. Phys.*, 21:232–237, 1950.

[WW93] H. Werner and H. Wengle. Large eddy simulation of turbulent flow around a cube in a plane channel. In F. Durst, R. Friedrich, B. Launder, U. Schumann, and J. Whitelaw, editors, *Selected Papers From the 8th Symp. on Turb. Shear Flows*, pages 155–168, New York, 1993. Springer.