EFFECTS OF NOISE ON ELLIPTIC BURSTERS

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ABSTRACT. Elliptic bursting arises from fast-slow systems and involves recurrent alternation between active phases of large amplitude oscillations and silent phases of small amplitude oscillations. This article is a rigorous analysis of elliptic bursting with and without noise. We first prove the existence of elliptic bursting solutions for a class of fast-slow systems without noise by establishing an invariant region for the return map of the solutions. For noisy elliptic bursters, the bursting patterns depend on random variations associated with delayed bifurcations. We provide an exact formulation of the duration of delay and analyze its mean and variance. The duration of the delay, and consequently the durations of active and silent phases, is shown to be closely related to the logarithm of a distance function that is nearly Gaussian and propotional to the amplitude of the noise. The treatment of noisy delayed bifurcation here is a general theory of delayed bifurcation valid for other systems involving delayed bifurcation as well, and is a continuation of the rigorous Shishkova-Neishtadt theory on delayed bifurcation or delay of stability loss.

1.INTRODUCTION

In several brain areas, neurons have been observed experimentally to engage in a rhythmic pattern of behavior referred to as elliptic bursting. In elliptic bursting, neuronal activity alternates between active phases, characterized by large amplitude oscillations, and quiescent phases, associated with oscillations of much smaller amplitudes (see Figures 2 and 3). Neuronal examples are given in the context of thalamic sleep rhythms and other neuronal systems in [6-8,17,23,28,39-40,52]. We

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study a model of elliptic bursting poposed by Rinzel [37] and considered previously both by numerical simulation and by fast-slow dissection in a singular limit [37,39,53].

The complications involved in such systems are related to a dynamical phenomenon known as delayed bifurcation or delay of stability loss, defined by Arnold [2]. Solutions stay close to a quasi-steady state as the $O(\varepsilon)$-slow variable passes through a threshold where linear stability is lost. Subsequently, after a substantial $O(1)$ delay, solutions jump away from quasi-steady state. These issues have been studied by many authors [1-5,9-13,15-19,25-27,29-36,42-49]. When a system can be reduced to a homogeneous system (i.e., zero is an obvious solution), the delay can be attributed to a simple contraction of solutions. But in general, more conditions are required for delay. In fact, in the case of slow passage through a simple eigenvalue bifurcation where contraction is also present, the amount of delay can be rather small and may vanish as the slowness goes to zero, as shown in examples by Lebovitz [26-27], Ahlers [1] and Kapila [19]. The delay in slow passage through a Hopf bifurcation is generically more significant for systems that are analytic in complex time, as shown by Shishkova [43], Neishtadt [31-36] and many others [3,10,15,43-49]. Even in this case, however, the amount of delay still relates to many factors, such as nearby singularities and, if external forcing is present, the difference between intrinsic and forcing frequencies [3,36,47-49]. When noise is added, numerical computations [3, 54] and asymptotic methods [22-23] suggest that the amount of delay is signficantly reduced. These results clarify why delay has not been observed in certain noisy environments [18].

When the delayed bifurcation is incorporated into a fast-slow system to model bursting phenomena, chaotic behavior is expected. In fact, for a similar system involving delayed bifurcation, Schecter [42] proved that the Poincaré map contains a Smale horseshoe, among other properties. A recent study of Kuske and Baer [23] introduced noise of Brownian motion type into an elliptic bursting system. Depending on the amplitude of the noise, it was found that there are regular patterns of alternations between a long active phase and a long silent phase, regular patterns of alternations between short active and silent phases, as well as irregular patterns of alternations of phases with various time durations (see in particularly Figure 3.2 (e-g) of Kuske and Baer [23]). When the noise amplitude is set to be extremely close to zero, the irregular patterns give way to a pattern that strongly resembles deterministic elliptic bursting. But even with a noise of quite small magnitude, the irregularity is significant. Kuske and Baer [22-23] determine that this irregularity follows from random variation in the delay of stability loss, based on an asymtptotic approximation of the probability density function for the state of the system in the silent phase and an asymptotic analysis of the effect of noise on transitions out of the active phase.

This article is a rigorous analysis of elliptic bursting with and without noise. We first prove the existence of elliptic bursting solutions for a class of fast-slow systems without noise by establishing an invariant region for the return map of the solutions. For noisy elliptic bursters, bursting patterns depend on random variations associated with delayed bifurcation. We provide an exact formulation of the duration of delay and analyze its mean and variance. The duration of the delay, and consequently the durations of active and silent phases, is shown to be closely related to the logarithm of a distance function that is nearly Gaussian and propotional to the amplitude of the noise. The treatment of noisy delayed bifurcation here is a general theory of delayed bifurcation valid for other systems involving delayed simple eigenvalue or delayed Hopf bifurcation as well, and is a continuation of the rigorous Shishkova-Neishtadt theory on delayed bifurcation or delay of stability loss [31-36,43].

The phenomenon to be discussed here is different from the chaotic behavior caused by different initial positions of the deterministic dynamics as in [42]. Rather, the irregular patterns are derived from solutions with the same initial position, as noise properties are varied. Further, solutions with different initial positions behave in similar ways.

The paper is organized in the following way. In Section 2, we state general assumptions on an elliptic bursting model without noise and state the existence result for deterministic bursting solutions. The proof of this result is presented in Section 3. In Sections 4 and 5, we consider the dynamical behavior of the system with noise and establish the relation between the bursting patterns and the amount of delay due to the slow passage through a Hopf bifurcation. The amount of delay will be random but is closely related to a normal distribution.

2. General assumptions and results on elliptic bursters .

Our assumptions on the elliptic bursting model are quite general. Following Rinzel [37] and Wang and Rinzel [53], assume the variables v (e.g., the voltage across a neuronal membrane), w (e.g., the activation of a fast ionic current through the membrane), and y (e.g., the activation of a slow ionic current) satisfy the differential equations

$$v' = f_1(v, w, y), \tag{2.1a}$$

$$w' = f_2(v, w, y), \tag{2.1b}$$

$$y' = \varepsilon g(v, w, y) \tag{2.1c}$$

where $0 < \varepsilon \ll 1$ and f_1, f_2, g are smooth (see (H4) below). The corresponding system with $\varepsilon = 0$ is called the fast subsystem (FS). Equation (2.1c) is called the slow equation (SE).

Assume for the system (2.1) that

(H1) There exists an interval $[y_\lambda, y_\rho]$ of y-values on which the set of equilibria of (FS) is a curve of the form $S = \{U_y \equiv (v_0(y), w_0(y), y) \,|\, y_\lambda \le y \le y_\rho\}$.

(H2) (FS) features a subcritical Hopf bifurcation along S, at $p_H = (v_H, w_H, y_H)$, $y_\lambda < y_H < y_\rho$, with a corresponding saddle-node of periodic orbits for $y = y_r \in (y_H, y_\rho)$. The outer periodic solutions, which we denote as $P_y(t)$ for each y, are stable, while the inner ones are unstable (see Figure 1). The family $P \equiv \{P_y(t) \,|\, y_h \leq y \leq y_r\}$ terminates at $y = y_h < y_H$ (possibly in a homoclinic orbit or in another Hopf bifurcation point). Our analysis will assume that trajectories do not enter the vicinity of y_h.

(H3) There exists a $y_R > y_r$ such that for $y \in (y_\lambda, y_R)$, the equilibrium curve S belongs in the region $\{g < 0\}$. Along each periodic orbit P_y, the motion of y follows an averaged equation derived from (SE). Specifically, we have $y(t) = y_0(t) + O(\varepsilon)$, with y_0 a solution of the averaged equation ${y_0}' = \varepsilon \hat{g}(y_0)$ defined later, and $\hat{g}(y) > 0$ for each $y \in [y_h, y_r]$, we further assume $\hat{g}(y) > 0$ for the inner periodic branch when y is near y_r, which will prevent canard phenomena, as discussed in Section 3.

Remark 2.1: Define the y-nullsurface $N \equiv \{(v, w, y) \,|\, g = 0\}$. This may intersect S at $y > y_R$, or not at all (as in [23]). In fact, the results below will hold if N intersects S at some y sufficiently far below y_H, as in [39-40]. We will comment specifically on this case in Remark 3.3 below.

We make some additional assumptions that are necessary for delayed bifurcation problems. These assumptions are satisfied by the FitzHugh-Nagumo equations and by other neural models under consideration [3,7-8,22-23,38-39,44-45].

(H4) The vector functions $f = (f_1, f_2) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ have analytic extensions for $|(v, w)| < \sigma_a$, $|y| < r_a$ in the complex plane for some $\sigma_a, r_a > 0$.

(H5) For each fixed $y \in (y_\lambda, y_\rho)$, the equilibrium curve $(v_0(y), w_0(y))$ is analytic in y for $|y| < r_a$.

Now, consider the variational equations of (FS) about $(v_0, w_0)(y)$, namely

$$z_t = f_{(v,w)}((v_0, w_0)(y), y)z \,, \tag{2.2}$$

a linear system with coefficients depending on the parameter y. Let $A(y) = f_{(v,w)}((v_0, w_0)(y), y)$.

(H6) Assume that two eigenvalues of $A(y)$, denoted by $\xi_1(y)$ and $\xi_2(y)$, are complex conjugate to each other i.e., $\xi_2(y) = \bar{\xi}_1(y)$ for each y on the real axis and $|y| < r_a$. Further, near the Hopf bifurcation point y_H, $Re\xi_j(y) < 0$ when $y > y_H$ and $Re\xi_j(y) > 0$ when $y < y_H$. To distinguish the two, we assume that $Im\xi_1(y_H) < 0$. We further assume a transverse crossing occurs, so that $-\frac{dRe\xi_j(y)}{dy} = a_2 > 0$ at y_H.

We define an elliptic bursting solution to be a trajectory that alternates between active phases spent in a certain neighborhood N_P of P, where it undergoes large amplitude oscillations, and silent phases spent in a certain neighborhood N_S of S, where it undergoes small amplitude oscillations. Examples are shown in Figures 2 and 3. Under the assumptions (H1)-(H6), we derive the existence of elliptic bursting solutions to (2.1).

Theorem 2.1 (Elliptic bursting).

a. *There exists* $\varepsilon_0 > 0$ *such that the flow induced Poincaré map for equation (2.1) possesses an invariant region* S_H*, consisting of elliptic bursting solutions, for* $0 < \varepsilon \leq \varepsilon_0$*. More specifically,* S_H *is a 2-dimensional ring-shaped invariant region containing* P_{y_H}*, and further* S_H *is an absorbing set.*

b. *Each loop time* T_ε*, that is the time for an elliptic bursting solution to undergo any complete loop from entry into* N_S*, to entry into* N_P*, to re-entry into* N_S*, can be calculated as* $T_\varepsilon = T_1 + T_2 + \eta(\varepsilon)$*, where* T_1, T_2 *are* $O(\frac{1}{\varepsilon})$ *times associated with passage through the silent and active phases, respectively (see equations (3.8), (3.15), below), and* $\lim_{\varepsilon \to 0^+} \varepsilon\eta(\varepsilon) = 0$.

c. *Fix any elliptic bursting solution* $(\hat{v}, \hat{w}, \hat{y})$ *and time* t_1 *such that* $(\hat{v}, \hat{w}, \hat{y})(t_1) \in S_H$*. When* $\varepsilon \leq \varepsilon_0$*, for any* $\delta > 0$ *sufficiently small and any solution* (v, w, y) *of equation (2.1) such that* $(v, w, y)(0) \in S_H$ *and therefore* $y(0) = \hat{y}(t_1) = y_H$*, there* *exists an* $M = M(\varepsilon_0) > 0$ *such that*

$$|y(t) - \hat{y}(t + t_1)| \leq \delta \tag{2.3}$$

for the time duration $0 \leq t \leq (\frac{\delta}{M\varepsilon|\ln\varepsilon|})(T_1 + T_2 + o(\frac{1}{\varepsilon}))$, *corresponding to an* $(\frac{\delta}{M\varepsilon|\ln\varepsilon|})$ *number of elliptic bursting cycles. Therefore, elliptic bursting solutions are at least metastable.*

3. Analysis of Deterministic elliptic bursters.

To understand the dynamics of equation (2.1), we construct neighborhoods N_S for the equilibrium curve S and N_P for the periodic family P. Let $M > 0$ be a constant, and let

$$D = \{(x, u, y) \,|\, x^2 + u^2 \leq (M\varepsilon)^2, |y| \leq 1\},$$

$$D_L = D \cap \{y = -1\},$$

$$D_S = D \cap \{x^2 + u^2 = (M\varepsilon)^2\},$$

$$D_R = D \cap \{y = 1\}.$$

Let E^3 denote $(v, w, y)-$phase space. The projection $\pi_y : E^3 \to E^1$ is given by $\pi_y(v, w, y) = y$. We define that $\phi : \mathbb{R}^3 \to E^3$ is a y-homeomorphism if ϕ is a homeomorphism and $\pi_y\phi(x_1, u_1, y_1) \leq \pi_y\phi(x_2, u_2, y_2)$ when $y_1 \leq y_2$.

(A) The steady branch and its dynamics.

The trajectories of the solutions near the steady branch S can be described as follows.

Proposition 3.1. *There exists* $\varepsilon_S > 0$ *and* $M(\varepsilon_S) > 0$ *such that for* $0 < \varepsilon \leq \varepsilon_S$, *there exists a y-homeomorphism* $\phi_S : D \to E^3$ *for which* $N_S \equiv \phi_S(D)$ *forms a neighborhood of the steady branch* S *with the following properties:*

a) $S \subset N_S$, *and the Hopf bifurcation point* y_H *corresponds to* $z = 0$ *at the center of the tube* D, *b)* $N_S \subset M_- \equiv \{g < 0\}$,

c) $\pi_y(\phi_S(D_R)) = y_\rho$,

d) if $\gamma(t_0) \in \phi_S((\partial D \setminus D_R) \cap \{z > 0\})$ *is on the boundary of* N_S *at* $y = y_0 > y_H$, *then* $\gamma(t)$ *enters and remains within* N_S *in forward time until it exits* N_S *with* $y = y^0 < y_H$ *at a time* $T_1 + O(\varepsilon)$, *where both* y^0 *and the time duration* T_1 *are functions of* y_0.

Proof of Proposition 3.1. We make the change of variables $\hat{x} = v - v_0(y)$, $\hat{u} = w - w_0(y)$, which translates the steady branch S to the origin of (FS) for each y, and then diagonalize the system, using $(\hat{x}, \hat{u}) = B(y, \varepsilon)(x, u)$. Near the steady branch, equation (2.1) thus becomes

$$x' = \xi_1(y)x + g_1(x, u, y) + \varepsilon h_1(y, \varepsilon), \tag{3.1a}$$

$$u' = \bar{\xi}_1(y)u + g_2(x, u, y) + \varepsilon h_2(y, \varepsilon), \tag{3.1b}$$

$$y' = \varepsilon g[(v_0(y), w_0(y)) + B(y, \varepsilon)(x, u), y] = -\varepsilon g_3(x, u, y, \varepsilon) \tag{3.1c}$$

where $B(y, \varepsilon)$ is a diagonalizing matrix , $u = \bar{x}$, $g_2 = \bar{g}_1$, and $h_2 = \bar{h}_1$, with

$$g_1 = O(\varepsilon)x + O(\varepsilon^2)u + O(|x|^2, |u|^2), \tag{3.2a}$$

$$g_2 = O(\varepsilon)u + O(\varepsilon^2)x + O(|x|^2, |u|^2), \tag{3.2b}$$

and $h_i = O(1)$. The different orders in equation (3.2) occur because when we diagonalize the right hand side of (3.1), a higher order off-diagonal term arises from differentiation of $B(y, \varepsilon)(x, u)$ with respect to y. For slow equation (3.1c), we have $0 < g_3 = g_4(y, \varepsilon) + o(|x| + |u|)$, which is positive when (x, u) is small by (H3). We introduce y as the new independent variable and change equation (3.1) into

$$\varepsilon x_y = -\lambda_1(y)x + \hat{G}_1(x, u, y) + \varepsilon H_1(y, \varepsilon), \tag{3.3a}$$

$$\varepsilon u_y = -\bar{\lambda}_1(y)u + \hat{G}_2(x, u, y) + \varepsilon H_2(y, \varepsilon). \tag{3.3b}$$

The higher order terms satisfy

$$\hat{G}_1 = O(\varepsilon)x + O(\varepsilon^2)u + O(|x|^2, |u|^2), \tag{3.4a}$$

$$\hat{G}_2 = O(\varepsilon)u + O(\varepsilon^2)x + O(|x|^2, |u|^2) \tag{3.4b}$$

and $H_i = O(1)$. By (H6), the eigenvalues satisfy $Re\lambda_j(y) < 0$ when $y > y_H$; $Re\lambda_j(y) > 0$ when $y < y_H$; and $Im\lambda_1(y_H) < 0$. Further, a transverse crossing occurs so that $-\frac{dRe\lambda_j(y)}{dy} = a_3 > 0$ at the Hopf bifurcation point $y = y_H$.

There are numerous discussions [1-5,9-13,15-19,25-36,42-49] on the behavior of solutions to equation (3.3). Most of the previous work considers how y increases past y_H, while in our case, y decreases past y_H. We keep the minus sign in front of λ_1 in equation (3.3) to preserve the consistency of notation with other related works. We summarize some relevant results in the following theorem.

Theorem 3.2 [32-33,44-45]. *Let $(x, u)(y, \varepsilon)$ be any family of solutions of equations (3.3-3.4) with initial conditions that satisfy $|(x, u)(y_i, \varepsilon)| \leq M_1\varepsilon$ for $y_i > y_H$ and some $M_1 > 0$. Then there exist $M = M(M_1) > 0, y_q = y_q(M_1, M) < y_H, \varepsilon_S = \varepsilon_S(M_1, M)$ such that*

$$|(x, u)(y, \varepsilon)| \leq M\varepsilon$$

whenever $y_i \geq y \geq y_q, 0 \leq \varepsilon \leq \varepsilon_S$. Further, if y_i is close enough to y_H, then y_q and y_i satisfy the relationship

$$\int_{y_i}^{y_q} Re\lambda_1(\tau)d\tau = 0. \tag{3.5}$$

Thus, we can simply choose $M_1 > 0$, find $M(M_1)$ and $\varepsilon_S(M, M_1)$ from Theorem 3.2, and for any $0 < \varepsilon \leq \varepsilon_S$ set $D = \{x^2 + u^2 \leq (M\varepsilon)^2, y_l \leq y \leq y_\rho\}$ where $y_l < y_H$ is the point satisfying

$$\int_{y_l}^{y_\rho} Re\lambda_1(\tau)d\tau = 0. \tag{3.6}$$

If (x, u, y) enters D at $y = y_0$, then it must exit D at $y = y^0$, where $y^0 < y_H$ is the point such that

$$\int_{y_0}^{y^0} Re\lambda_1(\tau)d\tau = 0. \tag{3.7}$$

The time duration can be calculated from the slow equation (3.1c),

$$\int_{y_0}^{y^0} \frac{1}{g(U_y, \varepsilon)}\, dy = \varepsilon T_1 + O(\varepsilon^2). \tag{3.8}$$

With these results in hand, the rest of the argument in Proposition 3.1 follows easily. In fact, there exists a y-diffeomorphism $\psi : D \to E^3$ such that under ψ, the slow equation has the canonical form

$$y' = -\varepsilon. \tag{3.9}$$

□

Remark 3.1: Fix a solution $(\tilde{x}, \tilde{u})(y, \varepsilon)$ described in Theorem 3.2, such that (3.5) holds, but with y_i as far away from y_H as possible. Let $(X, U) = (x, u) - (\tilde{x}, \tilde{u})(y, \varepsilon)$. This transforms equation (3.3) into a homogeneous system for (X, U), namely

$$\varepsilon X_y = -\lambda_1(y)X + G_1(X, U, y, \varepsilon), \tag{3.10a}$$

$$\varepsilon U_y = -\bar{\lambda}_1(y)U + G_2(X, U, y, \varepsilon) \tag{3.10b}$$

where

$$G_1 = O(\varepsilon)X + O(\varepsilon^2)U + O(|X|^2, |U|^2), \tag{3.11a}$$

$$G_2 = O(\varepsilon)U + O(\varepsilon^2)X + O(|X|^2, |U|^2). \tag{3.11b}$$

Since $U = \bar{X}$, we write equation (3.10) in the complex form

$$\varepsilon X_y = -\lambda_1(y)X + G_1(X, \bar{X}, y, \varepsilon), \tag{3.12}$$

where G_1 has the form $G_1 = \varepsilon a(y,\varepsilon)X + O(\varepsilon^2)\bar{X} + O(X^2, X\cdot\bar{X}, \bar{X}^2)$ and equation (3.12) has an analytic extension into the complex plane z,

$$\varepsilon X_z = -\lambda_1(z)X(z) + G_1(X(z), \bar{X}(\bar{z}), z, \varepsilon), \tag{3.13}$$

where $\bar{X}(\bar{z})$ is the analytic extension for $\bar{X}(\bar{y})$. This will be important in Section 5.

(B) The periodic branch and its dynamics.

The behavior of solutions near a family of periodic orbits, terminating at one end in a Hopf bifurcation and at the other end in a homoclinic bifurcation, was discussed in detail by Terman [50-51] (see also Rubin and Terman [40]), and we use similar ideas here.

Recall that each $P_y(t)$ is an asymptotically stable periodic solution. For each $y \in (y_h, y_r)$, we seek a compact neighborhood of $P_y(t)$ in $E^3 = (v,w,y)$ phase space. In particular, let

$$A = \{(x,u,y) : 1 - 2M\varepsilon \le x^2 + u^2 \le 1 + 2M\varepsilon, -1 \le y \le 1\},$$

$$A_R = \{(x,u,y) : 1 - 2M\varepsilon \le x^2 + u^2 \le 1 + 2M\varepsilon, y = 1\},$$

$$A_L = \{(x,u,y) : 1 - 2M\varepsilon \le x^2 + u^2 \le 1 + 2M\varepsilon, y = -1\},$$

$$A_S = \{(x,u,y) : x^2 + u^2 = 1 - 2M\varepsilon, \text{or}\, x^2 + u^2 = 1 + 2M\varepsilon, \quad -1 \le y \le 1\}.$$

Proposition 3.3. *There exists $\varepsilon_P > 0$ and $M(\varepsilon_P) > 0$ such that for $0 < \varepsilon \le \varepsilon_P$, there exists a y-homeomorphism $\phi_P : A \to E^3$ for which $N_P \equiv \phi_p(A)$ forms a neighborhood of the periodic branch P with the following properties:*

a) $P \subset N_P$ and the right knee of P at $y = y_r$ is at the right end of N_P corresponding to $z = 1$; that is, $\pi(\phi_P(A_R)) = y_r$,

b) $N_P \subset M_+ \equiv \{\hat{g} > 0\}$,

c) $\pi_y(\phi_P(A_L)) = y_\lambda$,

d) if $\gamma(t_0) \in \phi_P(\partial A \backslash A_R)$ is on the boundary of N_P at $y = y(t_0)$, then $\gamma(t)$ enters N_P in forward time and remains there until it exits at the right end $N_P \cap \{z = 1\}$

at time $T_2+t_0+O(\varepsilon)$, where the duration time T_2 is determined by the initial value $y(t_0)$.

Proof. The proposition follows from the stability properties of the periodic solutions of (FS). To apply the averaging method [41], we solve

$$V' = f_1(V,W,Y(t)), \tag{3.14a}$$

$$W' = f_2(V,W,Y(t)), \tag{3.14b}$$

$$Y' = \varepsilon\frac{1}{\tau(Y)}\int_0^{\tau(Y)} g(P_y(s),Y)\,ds \;\equiv \varepsilon\hat{g}(V,W,Y,\varepsilon), \tag{3.14c}$$

$$(V,W,Y)(t_0) = (v(t_0),w(t_0),y(t_0)) \tag{3.14d}$$

where $\tau(Y)$ is the period of the periodic solution $P_y(t)$ of (FS) with $y=Y$ and (v,w,y) denotes a solution to equation (2.1).

Fix $\varepsilon_P > 0$ sufficiently small such that the averaging theorem holds and let $M>0$. Assume that in equation (3.14d), $(v(t_0),w(t_0),y(t_0)) \in \phi_P(\partial A\setminus A_R)$. Let $(V_\varepsilon(t),W_\varepsilon(t),Y_\varepsilon(t))$ denote the corresponding solution to (3.14), describing averaged motions, where the motion of Y_ε is determined by the slow equation (3.14c), with $\hat{g}>0$ by (H3). Then $(v(t),w(t),y(t))$ stays $O(\varepsilon)$ near $(V_\varepsilon(t),W_\varepsilon(t),Y_\varepsilon(t))$ for an $O(\frac{1}{\varepsilon})$ time by the averaging theorem [41]. Further, we can choose $M(\varepsilon_P)>0$ such that for $0<\varepsilon\le\varepsilon_P$, $(V_\varepsilon(t),W_\varepsilon(t),Y_\varepsilon(t))$ in turn stays $O(\varepsilon)$ close to $P=\{P_{y(t)}\}$, since P is attracting. Therefore, the trajectory remains inside N_P and exits N_P at $z=1$, which is the right knee of P.

The time duration T_2 is basically the time span on the periodic branch from the time entering N_P at $y=y(t_0)$ to the time when the trajectory exits N_P at $y=y_r$. This is determined by the slow equation (3.14d) which yields the integral relation

$$\int_{y(t_0)}^{y_r}\frac{\tau(y)}{\int_0^{\tau(y)} g(P_y(s),y)\,ds}\,dy = \varepsilon T_2 + O(\varepsilon^2). \tag{3.15}$$

□

Remark 3.2: Using Fenichel coordinates [14] with N_P, we find that the averaged equation (3.14) can be reduced to the canonical form

$$r' = -(r-1), \tag{3.16a}$$

$$\theta' = c(y) > 0, \tag{3.16b}$$

$$y' = \varepsilon g_0(y) > 0. \tag{3.16c}$$

We shall use this form for the proofs below and for the study of the noisy case.

Proof of Theorem 2.1. We construct a region that is invariant under the Poincaré map induced by the flow of (2.1). Since the slow motion for y is oscillatory, rather than monotonic, along the periodic banch P, the Poincaré map needs to be carefully defined, using the averaged motion (3.14), for which the slow flow is monotonic.

To start, let $\varepsilon \leq \min\{\varepsilon_S, \varepsilon_P\}$, although ε may be decreased below, and let $M = \min\{M(\varepsilon_S), M(\varepsilon_P)\}$. In terms of the Fenichel coordinates (3.16), let

$$S_H = N_P \cap \{y = y_H\} = \{(r,\theta,y); |r-1| \leq M\varepsilon, \theta \in \mathbb{R}, y = y_H\}.$$

See Figure 2b. Let $\gamma(t)$ be any trajectory of equation (2.1) with $\gamma(0) \in S_H$. It is possible that $\gamma(t)$ does not exit S_H transversally, or $\gamma(t)$ exits to either the $y < y_H$ side or the $y > y_H$ side of S_H and then returns to S_H within an $O(1)$ time. Equation (3.16c), however, ensures that the averaged motion $\Gamma(t)$ of $\gamma(t)$ goes in the direction of increasing y. Thus, $\gamma(t)$ eventually passes S_H and in fact exits N_P through $\phi_P(A_R)$ at $y = y_r$.

We next need to ensure that $\gamma(t)$ enters N_S. Consider an extension of $\phi_P(A)$ given by

$$\tilde{\phi}_P = \{(v,w,y) \mid (v,w,y_r) \in \phi_P(A_R), y_r \leq y \leq y_s\}$$

for some $y_s > 0$. By the continuity of g, there exists of a choice of y_s, with $y_s - y_r = O(1)$, and a choice of $\varepsilon > 0$ sufficiently small in the definition of A such that

$$\frac{1}{\tau(y)}\int_0^{\tau(y)} g(v(t),w(t),y)\,dt > 0$$

for all $y \in [y_r, y_s]$. Since the only attractor for (FS) for $y > y_r$ is the branch of critical points S, the vector field of (2.1) points outward on $\partial\tilde{\phi}_P \cap \{y : y_r < y < y_s\}$; $\gamma(t)$ exits $\tilde{\phi}_P$ through this set for some $y < y_s$; and $\gamma(t)$ cannot re-enter $\tilde{\phi}_P$ without passing through N_P again. (Note that we do not need to consider canard-like passage along the unstable family of periodic orbits as long as the nullsurface $\{\hat{g} = 0\}$ is uniformly bounded away from P and the inner peiodic branch near the periodic saddle node, as indicated in (H3).)

The exponential attraction of $\gamma(t)$ to S, made explicit in equation (3.1), implies that $\gamma(t)$ enters N_S through the surface $\phi_S(D_S)$, in an $O(|\ln\varepsilon|)$ time after leaving $\tilde{\phi}_P$. Even if $y < y_r$ occurs before this entry, the family of unstable periodic orbits, the existence of which is given by (H1), acts as a separatrix that prevents $\gamma(t)$ from entering N_P instead.

Within $\tilde{\phi}_P$, the behavior of $\gamma(t)$ near the periodic saddle node is a subtle part of this analysis, as this is comparable to the similar cases of saddle nodes of critical points studied in [55].

In $\tilde{\phi}_P$, since $P = \{P_y\}$ depends smoothly on y, we can define a new polar coordinates (r,θ) near the periodic saddle node at $y = y_r$ such that $\{P_y\}$ and the inner periodic branch correspond to $\{r = r_+(y), 0 \le \theta \le 2\pi\}$ and $\{r = r_-(y), 0 \le \theta \le 2\pi\}$, respectively, with $r_\pm$ smooth in y. The periodic saddle node is at $r = r_+(y_r) = r_-(y_r) = r_0$.

Near $y = y_r$, under the new Fenichel coordinates, system (2.1) can be expressed by Taylor's Theorem as:

$$r' = a_1(\theta)(r - r_0) + a_2(\theta)(r - r_0)^2 + \sum_{n\geq 3} a_n(\theta)(r - r_0)^n + b_1(\theta)(y - y_r) +$$

(3.17a)

$$\sum_{j\geq 1, j+k\geq 2} b_{j,k}(\theta)(y - y_r)^j (r - r_0)^k,$$

(3.17b)

$$\theta' = C(r, \theta, y) > 0,$$

(3.17c)

$$y' = \varepsilon g(r, \theta, y)$$

where $a_j(\theta), b_j(\theta), b_{j,k}(\theta)$ are 2π-periodic. When $\varepsilon = 0$, both the outer periodic branch and the inner periodic branch are equilibria of equation (3.17a) for fixed y, independent of the angle θ. Therefore they satisfy the angle-averaged amplitude equation as well:

$$0 = r' = F(r, y) = \hat{a}_1(r - r_0) + \hat{a}_2(r - r_0)^2 + \sum_{n\geq 3} \hat{a}_n(r - r_0)^n +$$

(3.18)

$$\hat{b}_1(y - y_r) + \sum_{j\geq 1, j+k\geq 2} \hat{b}_{j,k}(y - y_r)^j (r - r_0)^k$$

where $\hat{a}_j = 1/(2\pi)\int_0^{2\pi} a_j(\theta)\, d\theta, j = 1, 2$, and $\hat{b}_1 = 1/(2\pi)\int_0^{2\pi} b_1(\theta)\, d\theta$ are in fact the 0^{th}-order terms of the Fourier series. Under the standard hypothesis [55-56] on the saddle node of quadratic type, we assume that $F_r\big|_{(r,y)=(r_0,y_r)} = \hat{a}_1 = 0$, $F_{rr}\big|_{(r,y)=(r_0,y_r)} = \hat{a}_2 < 0$ and $F_y\big|_{(r,y)=(r_0,y_r)} = \hat{b}_1 < 0$.

To establish the behavior of a solution (r, y) to the full equation (3.17) for $\varepsilon \to 0^+$, we consider the solution $(\hat{r}, \hat{y})$ of the averaged equation:

(3.19a)
$$\hat{r}' = \hat{a}_2(\hat{r} - r_0)^2 + \sum_{n\geq 3} \hat{a}_n(\hat{r} - r_0)^n + \hat{b}_1(\hat{y} - y_r) + \sum_{j\geq 1, j+k\geq 2} \hat{b}_{j,k}(\hat{y} - y_r)^j(\hat{r} - r_0)^k,$$

(3.19b)
$$\hat{y}' = \varepsilon\hat{g}(\hat{r}, \hat{y}) > 0,$$

(3.19c)
$$(\hat{r}, \hat{y})|_{t=0} = (r, y)|_{t=0}$$

where $\hat{g}(\hat{r}, \hat{y})$ is the averaged value of $g(\hat{r}, \theta, \hat{y})$, assumed to be positive in $\tilde{\phi}_P$ by (H3) to avoid the complication of canards. Since $(\hat{r}, \hat{y}) - (r, y) = O(\varepsilon)$ for $t \leq O(1/\varepsilon)$ by the averaging theorem [41], we consider the separation of $(\hat{r}, \hat{y})$ from the periodic saddle node instead. Since system (3.19) is exactly the case of a slow passage through a quadratic saddle node, the estimates of [55-56] imply that $\hat{y}$ and hence y can drift at most $O(\varepsilon^{2/3})$ beyond y_r before exit from $\tilde{\phi}_P$.

Once $\gamma(t)$ is inside of N_S, Proposition 3.1 implies that it exits N_S with $y = y^0 < y_H$ after time $T_1 + O(\varepsilon)$. The attraction of P implies that $\gamma(t)$ enters directly, in $O(|\ln\varepsilon|)$ time, into N_P, where Proposition 3.3 describes its behavior.

Thus, $\gamma(t)$ returns to S_H after a time $T_\varepsilon = T_1 + T_2 + o(1/\varepsilon)$. We therefore define the Poincaré map as the map Π from $\gamma(0)$ to $\gamma(T_\varepsilon)$, and S_H is an invariant set for Π. Further, since P is attracting, it can be shown that there exists a neighborhood S_G of S_H such that if $\gamma(0) \in S_G$, then $\gamma(t)$ will enter N_P in forward time and then, after one excursion of time $t \geq T_1 + T_2 + o(1/\varepsilon)$, enter S_H. Hence, S_H is also an absorbing set.

This establishes parts **a.** and **b.** of the theorem.

To obtain the result in **c.**, we only need to prove that there exists $M = M(\varepsilon_0)$ such that after $(\hat{v}, \hat{w}, \hat{y})$ complete a loop and return to S_H at $t_1 + T_\varepsilon = t_1 + (T_1 + T_2 + o(1/\varepsilon))$, the difference in $|y(t) - \hat{y}(t)| \leq M\varepsilon|\ln\varepsilon|, 0 \leq t \leq T_\varepsilon$. Then part **c.** follows by repeating the steps $\frac{\delta}{M\varepsilon|\ln\varepsilon|}$ times.

Accordingly, we calculate the difference in y and $\hat{y}$ during one loop. When both solutions belong to N_P, they are $O(\varepsilon)$ close to the corresponding averaged solutions in equation (3.16). We can verify directly any two solutions of equation (3.16) of same initial y value will maintain equal y-value. Therefore y and $\hat{y}$ remain $O(\varepsilon)$-close until they exit N_P.

When both solutions enter $\tilde{\phi}_P$, y and $\hat{y}$ are $O(\varepsilon)$-close to the averaged solutions in equation (3.19). Now it was shown in [56] that although the y- values of the solutions of equation (3.19) can drift away from y_r to $y_r + O(\varepsilon^{2/3})$ before exit from $\tilde{\phi}$, the solutions remain close together in a narrow strip of values of y of width $O(e^{-c/\varepsilon}), c > 0$, when they exit $\tilde{\phi}_P$. So y and $\hat{y}$ are still $O(\varepsilon)$-close to each other.

During the exponential decay to N_S near $y = y_r$ and the exponential growth from N_S to N_P near $y = y_l$, the total change of y can be calculated to be $O(\varepsilon|\ln\varepsilon|)$, see [44]. Equation (3.7) allows the computation of the y-value upon exit from N_S based on the y-values at entry into N_S. This implies that within N_S, the difference of y-values can grow by at most an $O(\varepsilon)$ amount.

In summary, for $\varepsilon_0 > 0$ sufficiently small and $0 < \varepsilon \le \varepsilon_0$, the errors accumulated in the neighborhoods described above imply that there exists an $M(\varepsilon_0) > 0$ such that $|y(t) - \hat{y}(t)| \le M\varepsilon|\ln\varepsilon|$ for $0 \le t \le T_\varepsilon$.

Remark 3.3: If the y-nullsurface N intersects S at some $y < y_H$, then the existence of elliptic bursting still follows as above. However, it is possible that $y' > 0$ may occur during passage through N_S (but below y_H). Thus, equation (3.7) no longer gives an estimate for the y-value at escape from N_S, and in simulations, a bursting trajectory may appeared to be pinned at some constant y-value during the transition from N_S to N_P, obscuring some of the delay in escape. This is shown in Figure 3.

Remark 3.4: The contraction behavior of solutions of equation (3.3) can be characterized by the following proposition; we will use a stochastic version of this in Section 5.

Proposition 3.4. *Assume that* $(x,u)_A$ *and* $(x,u)_B$ *are two solutions of equation (3.3) on* $y_1 < y < y_2$*, and* $|(x,u)_A(y)| \le M_2\varepsilon$*,* $|(x,u)_B(y)| \le M_2\varepsilon$ *whenever* $y_1 \le y \le y_2$*. Then there exist* $M_3 = M_3(M_2) > 0$*,* $\varepsilon_0 = \varepsilon_0(M_2) > 0$ *so that for* $0 < \varepsilon \le \varepsilon_0$*,*

$$(3.20)\quad \frac{1}{M_3} e^{\frac{-1}{\varepsilon}\int_{y_1}^{y_2} Re\lambda_1(\tau)d\tau} \le \frac{|(x,u)_A(y_2) - (x,u)_B(y_2)|}{|(x,u)_A(y_1) - (x,u)_B(y_1)|} \le M_3 e^{\frac{-1}{\varepsilon}\int_{y_1}^{y_2} Re\lambda_1(\tau)d\tau}.$$

Proof. See [44-45]. We note that there is no restriction on whether y_1 and y_2 are above or below y_H. $\square$

Note that the map Π is in general not a contraction. Therefore the existence and stability of a truly periodic solution remains open. In fact, the structure of the maximal attractor for the map can be quite interesting.

4. General assumptions and results on noisy elliptic bursters.

We now turn to the effect of a random force to the elliptic bursters. We consider the system

$$(4.1a)\qquad dv = f_1(v,w,y)dt + \varepsilon^2\sigma\hat{h}_1(y)dW(y),$$

$$(4.1b)\qquad dw = f_2(v,w,y)dt + \varepsilon^2\sigma\hat{h}_2(y)dW(y),$$

$$(4.1c)\qquad dy = \varepsilon g(v,w,y)dt$$

where the noisy term is modeled in similar terms by Baer et al. [3] and Kuske and Baer [22-23]. The magnitudes $\hat{h}_i$ are assumed to be positive. The Brownian motions $W(y)$ are based on the usual hypotheses:

(1) $W(y_1) - W(y_2)$ is Gaussian $N(0, \sqrt{|y_1 - y_2|})$,

(2) $W(y_1) - W(y_2)$ and $W(y_3) - W(y_4)$ are independent if intervals which ended at y_1,y_2 and y_3, y_4 are disjoint.

The parameter σ is ranged from exponentially small at $O(e^{-\frac{C}{\varepsilon}})$ to $O(1)$.

We consider the dynamic behavior of the elliptic burster under such a random noise. Particularly we are interested in the time durations T_1 on the steady branch and T_2 on the periodic branch. We see that motions near the periodic branch are not affected much by the noise because the periodic orbits are attracting, although the time duration T_2 does depend on the y value where the trajectory enters N_P. A short T_1 will be followed a short T_2 and a longer T_1 will be followed by a longer T_2. In brief, T_1 completely determines T_2, as we will show below. The duration T_1 spent inside N_S is therefore a key to understanding the patterns of the elliptic burster.

We study the time duration T_1 through a rigorous analysis of the distribution of the jumping point Y_j where the solution exits N_S and heads for N_P.

Theorem 4.1 (Main Theorem). *There exists ε_0 such that for $0 < \varepsilon \leq \varepsilon_0$, equation (4.1) possesses bursting solutions for which the loop time $T_\varepsilon(t)$ satisfies the relation $T_\varepsilon(t) = T_1 + T_2 + o(1/\varepsilon)$, where T_1, T_2 are the times spent inside N_S near the steady state curve S and inside N_P near the periodic orbit family P, respectively. The random variables $T_i, i = 1, 2$ are determined by the jump point Y_j through the relations*

$$\int_{y_r}^{Y_j} \frac{1}{g(U(y), \varepsilon)}\, dy = \varepsilon T_1 + O(\varepsilon^2) \tag{4.2}$$

and

$$\int_{Y_j}^{y_r} \frac{\tau(y)}{\int_0^{\tau(y)} g(P_y(s), y)\, ds}\, dy = \varepsilon T_2 + O(\varepsilon^2). \tag{4.3}$$

The jumping point Y_j is a random variable and is related to a random distance function Δ by the formula $\int_{y_H}^{Y_j} Re\lambda_1(y)\, dy = \varepsilon \ln(\Delta/M) + O(\varepsilon|\ln \varepsilon|)$. The distance *function Δ is a random variable dominated by a normal distribution $N(0, \delta_1)$ with mean $E(\Delta) = 0 + O(\sigma\varepsilon^3)$ and variance $V(\Delta) = \delta_1 + O(\sigma^2\varepsilon^3)$ where $\delta_1 = \sigma^2\varepsilon^{\frac{5}{2}}O(1)$.*

Therefore, only when $\sigma \leq O(e^{-C/\varepsilon})$ for large enough C is there a regular pattern of long bursts. When $O(1) \geq \sigma \geq O(\varepsilon^n), n \in \mathbb{N}$, there is a pattern of short bursts with a very small probability ($\leq O(\frac{e^{-C/\varepsilon}}{\varepsilon^n})$) of long bursts. When σ falls in range between $O(e^{-C/\varepsilon})$ and $O(\varepsilon^n)$, there will be a nontrivial distribution of bursts of different lengths.

Remark 4.1 To be more specific, when $\sigma \geq O(\varepsilon^n), n \in \mathbb{N}$, the points of exit from N_S are $Y_j = y_H - O(\varepsilon \ln(\varepsilon))$, located very near the bifurcation point Y_H, and we see a short period of large spikes in active phase as well as a short period of small oscillations in silent phase. When $\sigma \leq O(e^{-C/\varepsilon})$, $Y_j = y_H - O(1)$, and longer periods in each phase are exhibited. See Figures 4a and 4b. For the values of σ in between $O(e^{-C/\varepsilon})$ and $O(\varepsilon^n)$, the mixed patterns of different phase lengths are related to a distribution that is nearly Gaussian, as illustrated in Figure 4c. Note that we write $-O(\dots)$ here to emphasize the fact that the jump out of N_S occurs after y drops below y_H, in all cases. In the statement of the Theorem and in the proof below, we use the conventional notation and write $+O(\dots)$, even when we know that the higher order terms are negative.

5. Analysis of noisy delayed bifurcations and elliptic bursters.

We use the same transformations used to derive equation (3.12) from equation (2.1) to transform the equation (4.1) into a system

$$\varepsilon X_y = -\lambda_1(y)X + G_1(X, \bar{X}, y) + \varepsilon^2\sigma\hat{H}(y)dW, \tag{5.1}$$

where G_1 has the expression $G_1 = \varepsilon a(y, \varepsilon)X + O(\varepsilon^2)\bar{X} + O(X^2, X \cdot \bar{X}, \bar{X}^2)$.

We need some technical lemmas to start, proceeding initially in analogy to the treatment of the determinstic case in [44-45].

Lemma 5.1. *Let $M_1 > 0$ and let $X = X(y, \varepsilon)$ be a family of solutions of the system (5.1) with initial conditions at $y = y_i$ satisfying $|X|\big|_{y=y_i} \equiv \sqrt{E(X\bar{X})}\big|_{y=y_i} \leq M_1\varepsilon^2$ for $y_i > y_H$. Then there exist $M_2 = M_2(M_1)$ and $\varepsilon_0 = \varepsilon_0(M_1) > 0$ so that when $0 < \varepsilon \leq \varepsilon_0$,*

$$\sqrt{E(X\bar{X})} \leq M_2\varepsilon^{3/2} \tag{5.2}$$

for $y_i \geq y \geq y_H$.

Sketch of the proof. Lemma 5.1 for equation (5.1) without the random terms was shown in [44-45]. For the stochastic equation (5.1), the situation is similar. The integral formulation for equation (5.1) is

$$X(y) = X(y_i)e^{\frac{-1}{\varepsilon}\int_{y_i}^{y}\lambda_1(s)ds} + \int_{y_i}^{y} e^{\frac{-1}{\varepsilon}\int_{\tau}^{y}\lambda_1(s)ds}(G_1(X, \bar{X}, \tau)d\tau + \varepsilon^2\hat{H}(\tau)dW). \tag{5.3}$$

By using the new norm $|X| \equiv \sqrt{E(X\bar{X})}$ and following the stability theorem for stochastic differential equations in [21,24], we derive the estimate

$$|X(y)| \leq |X(y_i)| + \frac{C}{\min_y |Re\lambda_1(y)|}|\varepsilon^2 H(y)| \tag{5.4}$$

provided $Re\lambda_1(y) < 0$. As in [44-45], except for the definition of the norm, we distinguish two cases.

Case 1 corresponds to $y_H + \sqrt{\varepsilon} \leq y \leq y_i$, for which $Re\lambda_1(y) \leq -c_1\sqrt{\varepsilon}$. We use the inequality (5.4) to derive $|X| \leq O(\varepsilon^{3/2})$.

Case 2 corresponds to $y_H \leq y \leq y_H + \sqrt{\varepsilon}$. Here, we use equation (5.3) from $y_H + \sqrt{\varepsilon}$ to y_H and use the fact the interval is smaller than $\sqrt{\varepsilon}$ to get $|X| \leq O(\varepsilon^{3/2})$ as well.

Lemma 5.2. *Let $M_1 > 0$, let $y^i < y_H$ be any point below the Hopf bifurcation point $y = y_H$, and let $X = X(y, \varepsilon)$ be solutions of equation(5.1) with initial conditions at*

$y = y^i$ which satisfy

$$|X|\big|_{y=y^i} \le M_1\varepsilon^2 \tag{5.5}$$

for any $\varepsilon > 0$. Then there exist $\varepsilon_0 = \varepsilon_0(M_1)$, $M_2 = M_2(M_1)$ so that for $\varepsilon \le \varepsilon_0$,

$$|X| \le M_2\varepsilon^{3/2} \tag{5.6}$$

whenever $y_H \ge y \ge y^i$.

Proof. If we replace the variable y by $J \equiv 2y_H - y$, then Lemma 5.2 follows immediately by analogous arguments as in Lemma 5.1. □

We also see that the exponential growth property, specified in Proposition 3.4, remains valid in this stochastic case.

Proposition 5.3. *Assume that X_A and X_B are two solutions of (5.1) on $y_1 < y < y_2$, and $|X_A(y)| \le M_2\varepsilon$, $|X_B(y)| \le M_2\varepsilon$ for some $M_2, \varepsilon > 0$ whenever $y_1 \le y \le y_2$. Then there exist $M_3 = M_3(M_2)$, $\varepsilon_0 = \varepsilon_0(M_2)$ so that for $\varepsilon \le \varepsilon_0$,*

$$\frac{1}{M_3} e^{\frac{-1}{\varepsilon}\int_{y_1}^{y_2} Re\lambda_1(\tau)d\tau} \le \frac{|X_A(y_2) - X_B(y_2)|}{|X_A(y_1) - X_B(y_1)|} \le M_3 e^{\frac{-1}{\varepsilon}\int_{y_1}^{y_2} Re\lambda_1(\tau)d\tau}. \tag{5.7}$$

This again follows similarly to the deterministic case because $|X_A(y)| \le M_2\varepsilon$, $|X_B(y)| \le M_2\varepsilon$.

Proof of Theorem 4.1. Using the eigenvalue $\lambda_1(y)$ from (5.1), define y_l in analogy to the determinstic case by

$$\int_{y_r}^{y_l} Re\lambda_1(\tau)d\tau = 0\,.$$

Let $X_0(y)$ be the solution of equation (5.1) with the initial condition $X_0(y_r) = 0$ and $X^0(y)$ be the solution of equation (5.1) with the initial condition $X^0(y_l) = 0$.

These initial positions are at different sides of the bifurcation point y_H as indicated in Figure 2 in the determinstic case.

We observe from Lemma 5.1 and Lemma 5.2 that both $X_0(y)$ for $y_H \le y \le y_r$ and $X^0(y)$ for $y_l \le y \le y_H$ stay $O(\varepsilon^{3/2})$. Then from Lemma 5.3, the behavior of the solution $X_0(y)$ beyond the bifurcation point y_H depends upon the distance $X_0 - X^0$ at $y = y_H$; indeed, the jumping point Y_j can be determined by

$$\frac{1}{M_3} e^{\frac{-1}{\varepsilon}\int_{y_H}^{Y_j} Re\lambda_1(\tau)d\tau} \le \frac{|X_0(Y_j) - X^0(Y_j)|}{|X_0(y_H) - X^0(y_H)|} \le M_3 e^{\frac{-1}{\varepsilon}\int_{y_H}^{Y_j} Re\lambda_1(\tau)d\tau}. \tag{5.8}$$

We define the jumping point Y_j as the first time at which the trajectory reaches the boundary of N_S, such that for $y_H \ge y > Y_j$, $|X_0(Y_j) - X^0(Y_j)| < M\varepsilon + O(\varepsilon^{3/2})$ and at $y = Y_j$, $|X_0(Y_j) - X^0(Y_j)| = M\varepsilon + O(\varepsilon^{3/2})$, since $X^0(Y_j) = O(\varepsilon^{3/2})$ by Lemma 5.2. Therefore, from equation (5.8), we can determine Y_j by the equation

$$\int_{y_H}^{Y_j} Re\lambda_1(\tau)d\tau = \varepsilon \ln |X_0(y_H) - X^0(y_H)| + O(\varepsilon|\ln\varepsilon|)\,. \tag{5.9}$$

Since $Re\lambda_1(\tau) > 0$ for $y_l \le \tau \le y_H$, the point Y_j is uniquely determined.

When $|X_0(y_H) - X^0(y_H)| = O(\varepsilon^n), n \in \mathbb{N}$, the delay will not be significant, $Y_j - y_H = O(\varepsilon|\ln\varepsilon|)$. Only if

$$|X_0(y_H) - X^0(y_H)| = O(e^{-\frac{C}{\varepsilon}}),$$

then $Y_j - y_H = O(1)$. In fact, with the help of Proposition 5.3, it can be shown that all solutions of equation (5.1) with initial conditions $|X(y_r)| = O(\varepsilon)$ (i.e., $X(y_r) \in N_S$) jump near the point Y_j at $Y_j + O(\varepsilon)$.

Let us study the distance $\Delta \equiv X_0(y_H) - X^0(y_H)$ which is random under the random influence $W(y)$. The distance Δ and the solutions X_0, X^0 are shown in Figure 5.

Consider $\gamma(y)$ to be a differentiable sample path of $W(y)$. The solutions of equation (5.1) under $\gamma(y)$ can be solved for in the following way. Let $X(y,s,\varepsilon)$ denote the solutions of (5.1) with such a sample path, such that

$$\varepsilon X_y = -\lambda_1(y)X + G_1(X,\bar{X},y) + \varepsilon^2\sigma\hat{H}(y)\gamma'(y), \tag{5.10}$$

with initial conditions $X(y,s,\varepsilon)|_{y=s} = 0, y_l \le s \le y_r$.

If we let $\eta = \frac{\partial}{\partial s}X(y,s,\varepsilon)$, then

$$\varepsilon\frac{\partial}{\partial y}\eta(y,s,\varepsilon) = -\lambda_1(y)\eta + \frac{\partial G_1}{\partial X}\eta, \tag{5.11a}$$

$$\eta(y,s,\varepsilon)\big|_{y=s} = -\varepsilon\sigma\hat{H}(s)\gamma'(s). \tag{5.11b}$$

The solution for equation (5.11) can be expressed as

$$\eta(y_H,s,\varepsilon) = -\varepsilon\sigma\hat{H}(s)\gamma'(s)e^{\frac{-1}{\varepsilon}\int_s^{y_H}(\lambda_1(\tau)-\frac{\partial G_1}{\partial X})d\tau}. \tag{5.12}$$

Then we derive that

$$\Delta \equiv X(y_H,y_r,\varepsilon) - X(y_H,y_l,\varepsilon) = \int_{y_l}^{y_r}\eta(y_H,s,\varepsilon)ds =$$

$$= \int_{y_l}^{y_r} -\varepsilon\sigma\hat{H}(s)\gamma'(s)e^{\frac{-1}{\varepsilon}\int_s^{y_H}(\lambda_1(\tau)-\frac{\partial G_1}{\partial X})d\tau}ds$$

$$= \varepsilon\sigma\int_{y_l}^{y_r} -\hat{H}(s)e^{\frac{-1}{\varepsilon}\int_s^{y_H}(\lambda_1(\tau)-\frac{\partial G_1}{\partial X})d\tau}d\gamma(s). \tag{5.13}$$

Thus under noise $W(y)$, the distance Δ can be expressed as

$$\Delta = -\varepsilon\sigma\int_{y_l}^{y_r}\hat{H}(s)e^{\frac{-1}{\varepsilon}\int_s^{y_H}(\lambda_1(\tau)-\frac{\partial G_1}{\partial X})d\tau}dW(s).$$

From the estimate of G_1 given in equation (3.11) and the fact that $X(y,s,\varepsilon) = O(\varepsilon^{3/2})$, we observe that for $y_l \le y \le y_r$,

$$\frac{\partial G_1}{\partial X} = \varepsilon a(y,\varepsilon) + G_3(X,s,\varepsilon) = \varepsilon a(y_H,\varepsilon) + O(\varepsilon^{3/2}). \tag{5.14}$$

Here we use equation (5.2) to derive the second equality.

Thus, we compute

(5.15)

$$\Delta = -\varepsilon\sigma\int_{y_l}^{y_r} \hat{H}(s)e^{\frac{-1}{\varepsilon}\int_s^{y_H}(\lambda_1(\tau)-\varepsilon a(y_H,\varepsilon))d\tau}(1+O(\sqrt{\varepsilon}))dW(s) = \Delta_1 + O(\sigma\varepsilon^{3/2}).$$

The first part Δ_1 is Gaussian. We calculate the mean and variance of Δ_1

$$E(\Delta_1) = \varepsilon\sigma\int_{y_l}^{y_r} -\hat{H}(s)e^{\frac{-1}{\varepsilon}\int_s^{y_H}(\lambda_1(\tau)-\varepsilon a(y_H,\varepsilon))d\tau}E(dW(s)) = 0 \tag{5.16}$$

from the properties of the Brownian motion. Thus $E(\Delta) = O(\sigma\varepsilon^{3/2})$. For any partition on P on [a,b], say $P = \{a = s_0 < s_1 < s_2 < \cdots < s_n = b\}$, we have the general formula that

$$E\left[\int_a^b f(s)dW(s)\overline{\int_a^b f(s)dW(s)}\right]$$

$$= \lim_{||P||\to 0} E[\Sigma_{i,j}f(s_i)(W(s_i)-W(s_{i-1}))\overline{f(s_j)}(W(s_j)-W(s_{j-1}))]. \tag{5.17}$$

Since $E[(W(s_i)-W(s_{i-1}))(W(s_j)-W(s_{j-1}))] = 0$ for $i \neq j$ due to independence and $E[(W(s_i)-W(s_{i-1}))(W(s_i)-W(s_{i-1}))] = |s_i - s_{i-1}|$, we have

(5.18)

$$E(\int_a^b f(s)dW(s)\overline{\int_a^b f(s)dW(s)}) = \lim_{||P||\to 0} \Sigma_i|f(s_i)|^2E(W(s_i)-W(s_{i-1}))^2 = \int_a^b |f(s)|^2ds.$$

The last equality (5.18) is known as Ito's lemma [20], derived from the properties of Brownian motion as stated in hypotheses (1), (2) in Section 4.

Thus, applying (5.18) to our situation, we get

$$E(\Delta_1\bar{\Delta}_1) =$$

$$\varepsilon^2\sigma^2 E\left(\int_{y_l}^{y_r}\hat{H}(s)e^{\frac{-1}{\varepsilon}\int_s^{y_H}(\lambda_1(\tau)-\varepsilon a(y_H,\varepsilon))d\tau}dW(s)\overline{\int_{y_l}^{y_r}\hat{H}(s)e^{\frac{-1}{\varepsilon}\int_s^{y_H}(\lambda_1(\tau)-\varepsilon a(y_H,\varepsilon))d\tau}dW(s)}\right)$$

$$= \varepsilon^2\sigma^2\int_{y_l}^{y_r}|\hat{H}(s)e^{\frac{-1}{\varepsilon}\int_s^{y_H}(\lambda_1(\tau)-\varepsilon a(y_H,\varepsilon))d\tau}|^2 ds$$

(5.19)

$$= \varepsilon^2\sigma^2\int_{y_l}^{y_r}|\hat{H}(s)|^2 e^{\frac{-1}{\varepsilon}\int_s^{y_H}(2Re\lambda_1(\tau)-2a(y_H,\varepsilon))d\tau}ds.$$

From the assumption of transversal eigenvalue crossing at the Hopf bifurcation, stated in (H6) in Section 2, we observe that near $s = y_H$,

$$Re\lambda_1(s) - \varepsilon a(y_H,\varepsilon) = -a_3(s-y_H) + O(s-y_H)^2 - \varepsilon a(y_H,\varepsilon),$$

for some $a_3 > 0$ and

$$\frac{-1}{\varepsilon}\int_s^{y_H}(2Re\lambda_1(\tau) - 2\varepsilon a(y_H,\varepsilon))\,d\tau = \frac{1}{\varepsilon}(-2\varepsilon a(y_H,\varepsilon)(s-y_H) - a_3(s-y_H)^2 + O((s-y_H)^3)).$$

Thus the distance function $E(\Delta_1\bar{\Delta}_1)$ in equation(5.19) can be bounded using the integral

$$\int_{y_l}^{y_r} e^{\frac{-a_3}{\varepsilon}(s-y_H)^2}ds = O(\sqrt{\varepsilon}).$$

More specifically, from a direct calculation [49], we can estimate the magnitude of the integral term in (5.19), as $\varepsilon \to 0^+$, by

$$C_1\sqrt{\varepsilon} \le \int_{y_l}^{y_r}|\hat{H}(s)|^2 e^{\frac{-1}{\varepsilon}\int_s^{y_H}(2Re\lambda_1(\tau)-2\varepsilon a(y_H,\varepsilon))d\tau}ds$$

(5.20)
$$= \int_{y_l}^{y_r}|\hat{H}(s)|^2 e^{\frac{1}{\varepsilon}(-2\varepsilon a(y_H,\varepsilon)(s-y_H)-a_3(s-y_H)^2+O((s-y_H)^3))}ds \le C_2\sqrt{\varepsilon}.$$

Therefore,

$$C_1\varepsilon^{5/2}\sigma^2 < E(\Delta\bar{\Delta}) = E(\Delta_1\bar{\Delta}_1) + O(\varepsilon^3\sigma^2) < C_2\varepsilon^{5/2}\sigma^2. \tag{5.21}$$

Further, using (5.15), (5.16), and (5.21), we bound the variance of Δ by

$$C_4\varepsilon^{5/2}\sigma^2 < E(|\Delta - E\Delta|^2) = E(\Delta\bar{\Delta}) - |E(\Delta)|^2 < C_3\varepsilon^{5/2}\sigma^2.$$

The rest of Theorem 4.1 follows from the properties of Δ. The time duration T_1 is obtained as the time of passage inside N_S from $y = y_r$ down to $y = Y_j$, under the flow of the slow equation (4.1c). This yields equation (4.2).

When $\sigma = O(\varepsilon^n)$, the first standard deviation of the distribution of the random variable Δ lies in a strip of width $\varepsilon^{n+5/4}$. In this case, the amount of delay will be $O(\varepsilon|\ln\varepsilon|)$, with a probability $O(e^{-\frac{C}{\varepsilon}})$ to have a significant $O(1)$ size delay. When $\sigma = O(e^{-\frac{C}{\varepsilon}})$, the delay will be significant almost surely. For the values in between, there will be nontrivial distributions of different amount of delays which are responsible for different patterns.

Remark 5.1: The treatment here is quite general and the analysis and result are valid both for delayed simple eigenvalue bifurcations as well as for delayed Hopf bifurcations, since the imaginary part of the eigenvalue $\lambda_1(y)$ is not contributing here.

The situation with the periodic branch is simpler due to the fact the periodic solutions of FS are orbitally stable. By using the Fenichel coordinates [14], equation (4.1) inside N_P can be reduced to a perturbation to equation (3.16),

$$dr = -(r-1)dt + \varepsilon^2\hat{h}_3(y,\varepsilon)dW(y), \tag{5.22a}$$

$$d\theta = c(y)dt + \varepsilon^2\hat{h}_4(y,\varepsilon)dW(y), \tag{5.22b}$$

$$y' = \varepsilon g_0(y) > 0. \tag{5.22c}$$

We can show easily that the solution $(R,\Theta,Y)(t)$ of the stochastic equation equation (5.22) and the solution $(r,\theta,y)(t)$ of equation (3.16), with the same initial condition at $t = t_0$, satisfy the relations

$$E(R, \Theta, Y) = (r, \theta, y), \tag{5.23}$$

and

$$E|(R, \Theta, Y) - (r, \theta, y)|^2 = O(\varepsilon^2). \tag{5.24}$$

Thus the solutions (R, Θ, Y) will remain within N_P and only exit when $y = y_r + O(\varepsilon |\ln \varepsilon|)$, the same as for (r, θ, y). Therefore the time duration T_2 is obtained as the passage time from $y = Y_j$ to $y = y_r$ inside N_P, using the slow motion for the averaged form of system (5.22). But by construction, this is equivalent to the passage time computed from the averaged form of system (4.1), which yields (4.3). Finally, the passages from N_S to N_P and from N_P to N_S are to leading order identical to the determinstic case.

Corollary. *Any solution (v, w, y) of equation (4.1) with its initial position in N_P or N_S is an elliptic bursting solution as described in Theorem 4.1.*

Proof. If $(v(0), w(0), y(0)) \in N_P$, then the corresponding solution $(v(t), w(t), y(t))$ of (4.1) enters N_S. Thus, without loss of generality, let the complex solution $X(y)$ correspond to (v, w, y), a solution of (4.1) with $(v(0), w(0), y(0)) \in N_S$. The point where the solution $X(y)$ will exit N_S is determined by the distance function

$$\Delta_2 = X(y_H) - X^0(y_H) = X(y_H) - X_0(y_H) + \Delta. \tag{5.25}$$

Now, by Proposition 5.3 and in particular equation (5.7), there exists a constant $M > 0$ such that

$$|X(y_H) - X_0(y_H)| \leq M|X(y_r) - X_0(y_r)|e^{\frac{-1}{\varepsilon}\int_{y_r}^{y_H} Re\lambda_1(\tau)\tau} \leq M\varepsilon(e^{-\frac{C}{\varepsilon}}),$$

we derive

$$\Delta_2 = \Delta + O(\varepsilon e^{-\frac{C}{\varepsilon}}). \tag{5.26}$$

The jumping point Y_j for any solution will the same one as in Theorem 4.1, up to a $O(\varepsilon|\ln\varepsilon|)$ error.

Remark 5.2: One additional consequence of the results in Sections 4 and 5 is that when an elliptic burst trajectory finishes one loop and restarts from y_r, the next loop will be independent from the previous one, but with its passage time through N_S determined by the same distribution.

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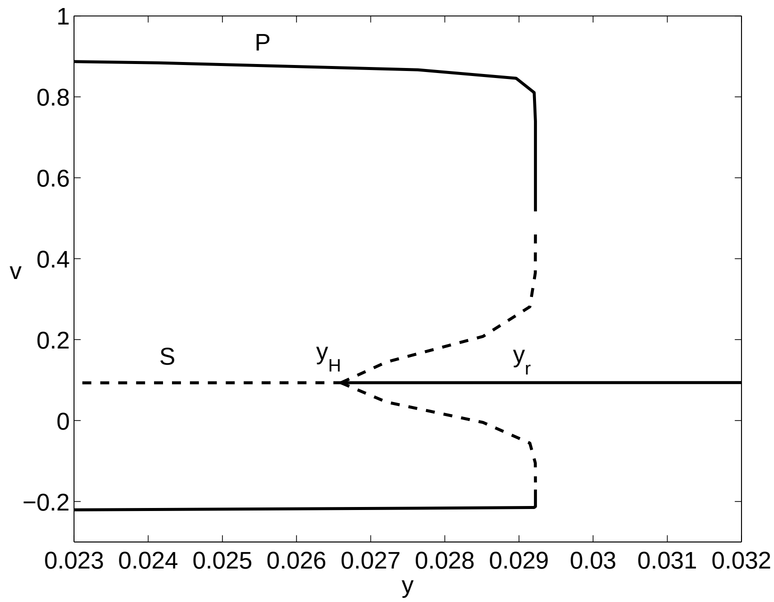

Figure 1. Bifurcation diagram for the fast subsystem (FS) of equation (2.1). This particular example was generated numerically from the Wu-Baer model for dendritic spine activity, as discussed in [23] (with parameters given in Figure 3.1 of [23]).

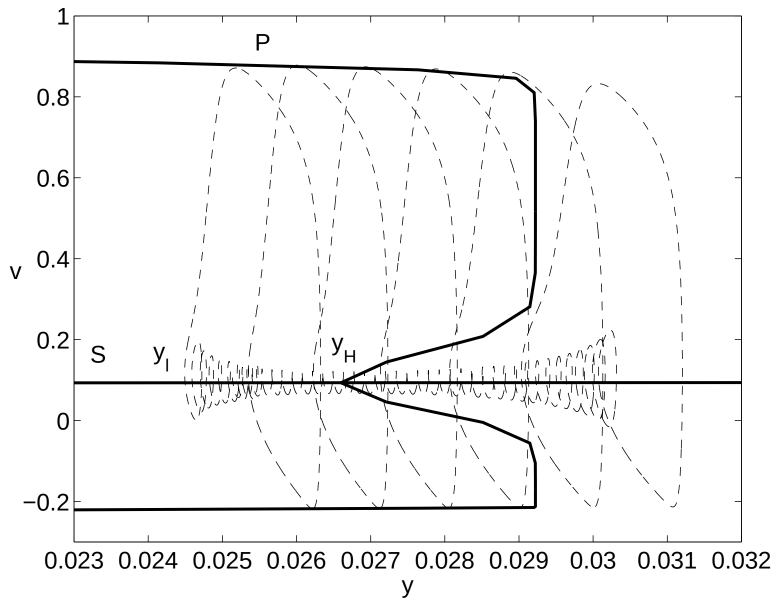

Figure 2a. Deterministic elliptic bursting in the Wu-Baer model with parameters from [23] but $\varepsilon = .003$. Note that the y-nullsurface does not intersect S in this model.

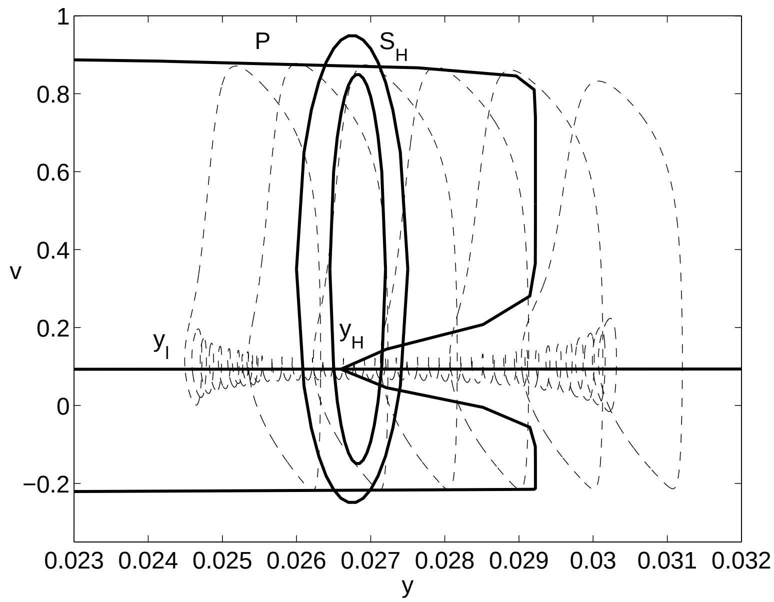

Figure 2b. The invariant region S_H.

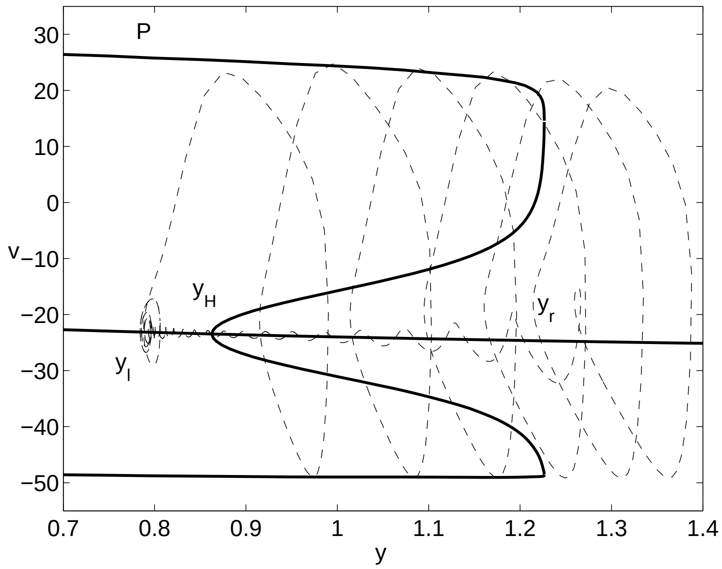

Figure 3. Deterministic elliptic bursting in a model for neuronal dynamics from [39], with parameters given in [40] except $\mu = 0.015$, which affects the location where the y-nullsurface intersects S. In this model, this intersection occurs below y_H, such that $y' > 0$ occurs during escape from S near $y = y_l$.

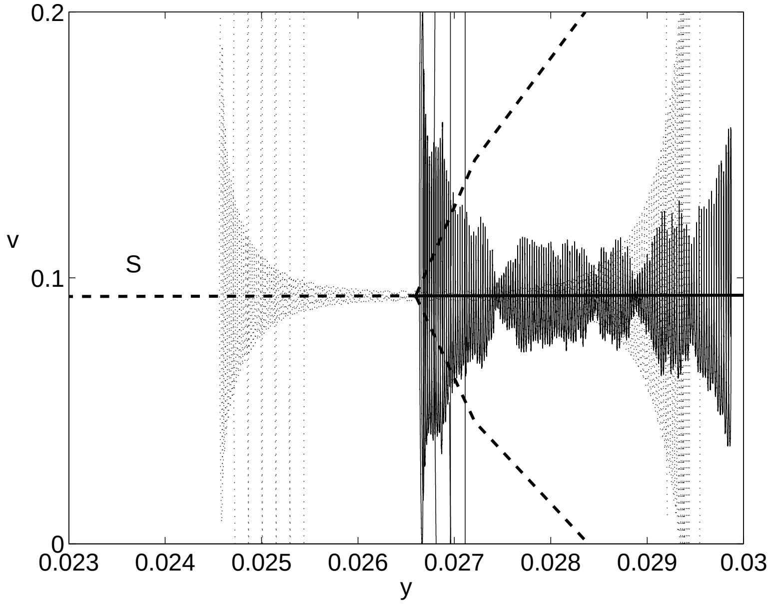

Figure 4a. Short duration noisy delay when $\sigma = O(\varepsilon^n)$. The solid curve shows a trajectory that jumps up extremely close to y_H with noise $\delta = \varepsilon = .0005$ in the Wu-Baer model, while the dotted curve shows a deterministic trajectory for the same parameter values. The dashed curves are the bifurcation curves for (FS); note that this figure is zoomed in, in the neighborhood of y_H, relative to Figures 1 and 2.

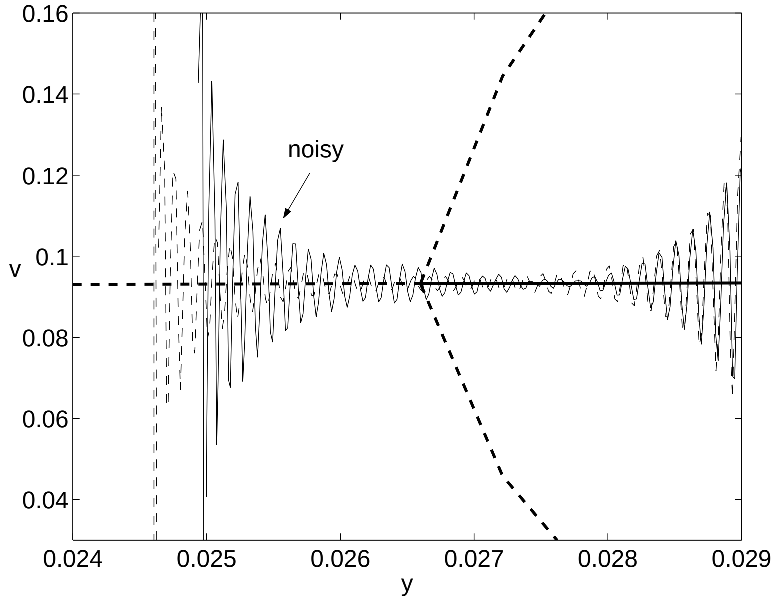

Figure 4b. Long duration noisy delay when $\sigma = O(\varepsilon^{-\frac{c}{\varepsilon}})$. The solid curve shows a trajectory that has a long duration noisy delay beyond y_H with noise $\delta = 5 \times 10^{-5}$ and $\varepsilon = .0005$ in the Wu-Baer model, while the dotted curve shows a deterministic trajectory for the same parameter values for comparison. The dashed curves are the bifurcation curves for (FS); The equations were solved with Euler's method with a time step of $dt = .02$; the values of (v, y) were plotted once every 3000 time steps for this figure.

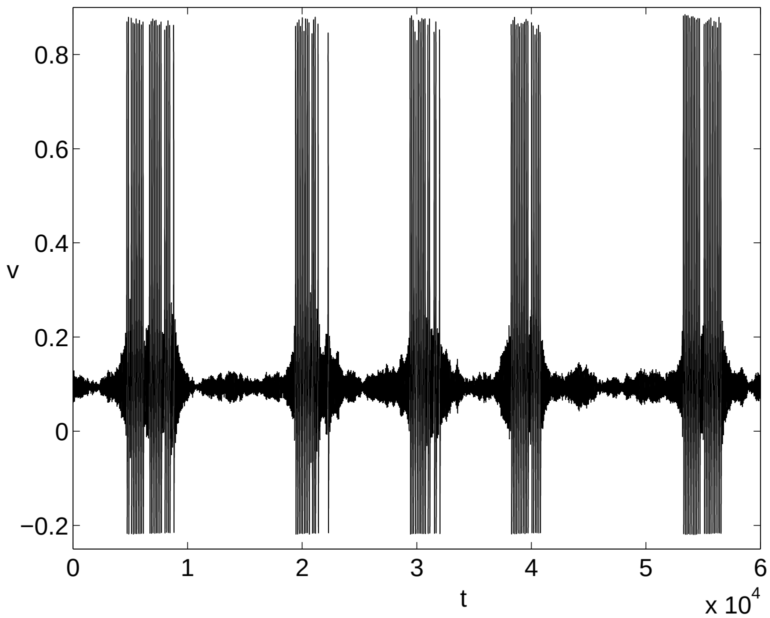

Figure 4c. A mixture of long and short delays and burstings. The curve, a trajectory in v versus t, has a mixture of long and short times in N_S for the Wu-Baer model with $\varepsilon = .0005$, noise $\delta = .00075$. The equations were solved with Euler's method with a time step of $dt = .05$; the values of (v, y) were plotted once every 5 time steps for this figure.

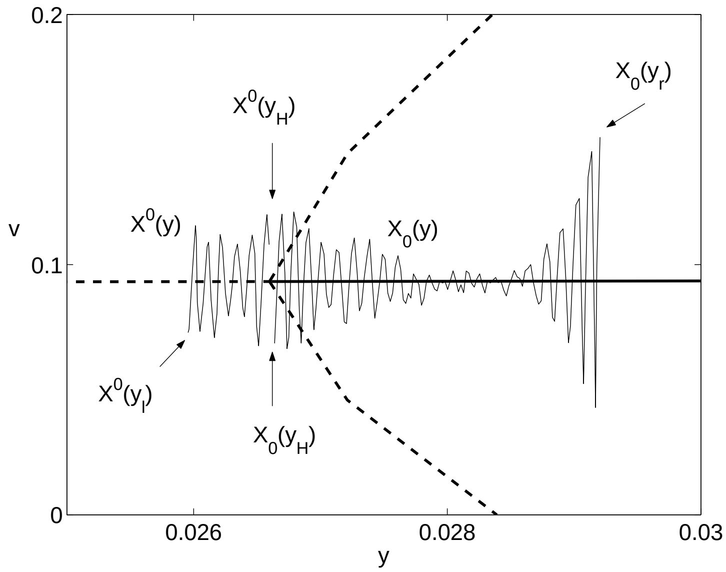

Figure 5: The distance function $\Delta = X_0(y_H) - X^0(y_H)$. These trajectories were computed numerically from the Wu-Baer model with $\delta = \varepsilon = .0005$. The equations were solved with Euler's method with a time step of $dt = .02$; the values of (v, y) were plotted once every 3000 time steps for this figure.