Linear Algebra Preliminary Exam
August 2015

Problem 1 *Let V be a finite dimensional vector space. Prove that every linearly independent set of V can be extended to a basis for V.*

Problem 2 *Let A, B, C be $n \times n$ matrices satisfying $AB = BA$. Show that*

$$\det(A + BC) = \det(A + CB).$$

Problem 3 *Prove that for any $n \times n$ complex matrix A,*

$$|\operatorname{tr}(A^* A)| \leq n \|A\|^2$$

where $\operatorname{tr}(A^* A)$ is the trace of $A^* A$ and $\|A\|$ is the operator norm of A defined by

$$\|A\| = \sup_{x \in \mathbb{C}^n,\, x \neq 0} \frac{|Ax|}{|x|}.$$

Problem 4 *Let P_4 be the vector space of polynomials with real coefficients of orders at most 3. Consider the linear map*

$$T : p(t) \mapsto p(t+1) - p(t).$$

a. Find all eigenvalues and corresponding eigenspaces of T.
b. Find the Jordan canonical form of T.

Problem 5 *Let $A(t)$ be a differentiable $n \times n$ matrix valued function. Is it always true that*

$$\frac{d}{dt} A^2(t) = 2A(t) \frac{dA(t)}{dt}?$$

Prove it or provide a counter example.

Problem 6 *Prove that if A is an invertible $n \times n$ matrix with integer entries then A^{-1} has integer entries if and only if* $\det(A) = \pm 1$.