Problem 1 Let $V$ be a finite dimensional vector space. Prove that every linearly independent set of $V$ can be extended to a basis for $V$.

Problem 2 Let $A, B, C$ be $n \times n$ matrices satisfying $AB = BA$. Show that

$$\det (A + BC) = \det (A + CB).$$

Problem 3 Prove that for any $n \times n$ complex matrix $A$,

$$|\text{tr} (A^*A)| \leq n \|A\|^2$$

where $\text{tr} (A^*A)$ is the trace of $A^*A$ and $\|A\|$ is the operator norm of $A$ defined by

$$\|A\| = \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{|Ax|}{|x|}.$$

Problem 4 Let $P_3$ be the vector space of polynomials with real coefficients of orders at most 3. Consider the linear map

$$T : p(t) \mapsto p(t+1) - p(t).$$

a. Find all eigenvalues and corresponding eigenspaces of $T$.

b. Find the Jordan canonical form of $T$.

Problem 5 Let $A(t)$ be a differentiable $n \times n$ matrix valued function. Is it always true that

$$\frac{d}{dt} A^2 (t) = 2A (t) \frac{dA (t)}{dt}?$$

Prove it or provide a counter example.

Problem 6 Prove that if $A$ is an invertible $n \times n$ matrix with integer entries then $A^{-1}$ has integer entries if and only if $\det(A) = \pm 1$. 