Linear Algebra Preliminary Exam
April 2015

Problem 1 *Let* Π *be the plane in* $\mathbb{R}^3$ *defined by*

$$3x + 2y + z = 0.$$

Let T *be the reflection in* $\mathbb{R}^3$ *about the plane* Π*. Find the matrix representation of* T*.*

Problem 2 *Suppose* $A \neq 0$ *is a nilpotent complex matrix. Prove that* $A + A^*$ *is not nilpotent.*

Problem 3 *Let* A *be a square matrix with characteristic polynomial*

$$(t+1)^2 (t-1)^3 \left(t^2+1\right)^2.$$

and minimal polynomial

$$(t+1)(t-1)^2 \left(t^2+1\right).$$

Find Jordan canonical forms of A*.*

Problem 4 *Consider a* **flag** *of linear subspaces* $\mathbf{F} = (\{0\} \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_n = \mathbb{C}^n)$*. Let* $T : \mathbb{C}^n \to \mathbb{C}^n$ *be a linear transformation such that for every* $i = 1, \ldots, n$, $T(F_i) = F_i$*. (Hint: Note that* $\dim F_k = k$ *for* $1 \leq k \leq n$*.)*
(a) Show that there is a basis $B = \{b_1, \ldots, b_n\}$ *for* $\mathbb{C}^n$ *with the following property: For each* $k = 1, \ldots, n$*, the set* $\{b_1, \ldots, b_k\}$ *is a basis for* F_k*.*
(b) Show that T *is represented by an upper-triangular matrix in this basis.*

Problem 5 *For two square matrices the Lie bracket* $[A, B]$ *is defined as*

$$[A, B] = AB - BA.$$

Let $\mathfrak{sl}(2, \mathbb{C})$ *denote the vector space of* 2×2 *complex matrices with trace* 0*.*
(a) Show that $\mathfrak{sl}(2, \mathbb{C})$ *is a* 3 *dimensional complex vector space and is closed under the Lie bracket.*
(b) Show that for any fixed matrix $M \in \mathfrak{sl}(2, \mathbb{C})$*, the map*

$$\mathbf{ad(M)} : A \mapsto [M, A]$$

is linear.
(c) Let $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$*. Find the eigenvalues and Jordan form of the linear operator* $\mathbf{ad(E)}$*. (Hint:* $\mathbf{ad(E)}$ *is represented as a* 3×3 *matrix.)*

Problem 6 *Let A be a 3×3 Hermitian matrix with eigenvalues $\gamma_1 \le \gamma_2 \le \gamma_3$. Let B be the 2×2 matrix which is the upper left corner entries of A. Let $\mu_1 \le \mu_2$ be the eigenvalues of B. Use Min-max theorem about eigenvalues to prove that these eigenvalues satisfy the interlacing inequalities:*

$$\gamma_1 \le \mu_1 \le \gamma_2 \le \mu_2 \le \gamma_3.$$

Hint: We recall Min-max theorem: Let A be an $n \times n$ Hermitian matrix with eigenvalues $\gamma_1 \le \cdots \le \gamma_n$ (we know all the eigenvalues are real because A is Hermitian). Then for each $1 \le k \le n$,

$$\begin{aligned}\gamma_k &= \max_{U \subset \mathbb{C}^n \textit{ is a subspace with} \dim(U) = n-k+1} \left\{ \min_{0 \ne x \in U} \frac{(x, Ax)}{(x, x)} \right\} \\ &= \min_{U \subset \mathbb{C}^n \textit{ is a subspace with} \dim(U) = k} \left\{ \max_{0 \ne x \in U} \frac{(x, Ax)}{(x, x)} \right\}.\end{aligned}$$