PRELIMINARY EXAM APRIL 2015

Problem 1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^2(y^4+2x)}{x^2+y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is differentiable at $(0,0)$.

Problem 2. A graph of a mapping $f : X \to Y$ is defined as

$$G_f = \{(x,y) \in X \times Y : \, y = f(x)\}.$$

Prove that if X is a metric space and Y is a compact metric space, then a map $f : X \to Y$ is continuous if and only if G_f is a closed subset of $X \times Y$.

Problem 3. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^\alpha}{n(n+x^2)}$

(a) converges uniformly on $[0,\infty)$ when $\alpha = 1$,
(b) converges pointwise but not uniformly on $[0,\infty)$ when $\alpha = 2$.

Problem 4. Find the distance between the two ellipsoids

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1$$

and

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 2,$$

where $a_1 > a_2 > \cdots > a_n > 0$.

Problem 5. Let $Q : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ be a linear mapping such that $Qf \geq 0$ whenever $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(0) = 0$ and $f(x) \geq 0$ in a neighborhood of 0. Prove that there are real numbers a_{ij}, b_i and c such that

$$Qf = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) + \sum_{i=1}^{n} b_i \frac{\partial f}{\partial x_i}(0) + cf(0).$$

Hint: *Estimate the Taylor reminder.*

Problem 6. Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $f \in C^\infty(\mathbb{R}^2)$. Suppose that $f(x,y) = 1$ for all $(x,y) \in \partial D$. Prove that

$$\iint_D \left(2f(x,y) + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \right) dA = 2\pi.$$