Problem 1. Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
\[
f(x,y) = \begin{cases} 
\frac{x^2(y^4 + 2x)}{x^2 + y^4} & \text{if } (x,y) \neq (0,0); \\
0 & \text{if } (x,y) = (0,0).
\end{cases}
\]
Show that \( f \) is differentiable at \((0,0)\).

Problem 2. A graph of a mapping \( f : X \to Y \) is defined as
\[ G_f = \{(x,y) \in X \times Y : y = f(x)\} \].
Prove that if \( X \) is a metric space and \( Y \) is a compact metric space, then a map \( f : X \to Y \) is continuous if and only if \( G_f \) is a closed subset of \( X \times Y \).

Problem 3. Prove that the series
\[
\sum_{n=1}^{\infty} \frac{x^n}{n(n+x^2)}
\]
(a) converges uniformly on \([0,\infty)\) when \( \alpha = 1 \),
(b) converges pointwise but not uniformly on \([0,\infty)\) when \( \alpha = 2 \).

Problem 4. Find the distance between the two ellipsoids
\[
\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1
\]
and
\[
\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 2,
\]
where \( a_1 > a_2 > \cdots > a_n > 0 \).

Problem 5. Let \( Q : C^\infty(\mathbb{R}^n) \to \mathbb{R} \) be a linear mapping such that \( Qf \geq 0 \) whenever \( f \in C^\infty(\mathbb{R}^n) \) satisfies \( f(0) = 0 \) and \( f(x) \geq 0 \) in a neighborhood of 0. Prove that there are real numbers \( a_{ij}, b_i \) and \( c \) such that
\[
Qf = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) + \sum_{i=1}^{n} b_i \frac{\partial f}{\partial x_i}(0) + cf(0).
\]
Hint: Estimate the Taylor remainder.

Problem 6. Let \( D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \) and \( f \in C^\infty(\mathbb{R}^2) \). Suppose that \( f(x,y) = 1 \) for all \((x,y) \in \partial D\). Prove that
\[
\iint_D \left( 2f(x,y) + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \, dA = 2\pi.
\]