Preliminary Examination in Mathematics
August 15th 2014
ID number:

Twenty points per question
The best six questions will count

Question 1

Let $\alpha \in \mathbb{R}$.
Let $f_\alpha : \mathbb{R}^2 \to \mathbb{R}$ be given by the formulas:

$$f_\alpha(0,0) = 0,$$

$$f_\alpha(x,y) = \frac{x^4 + y^4}{(x^2+y^2)^\alpha}, \text{ for any } (x,y) \in \mathbb{R}^2 - \{(0,0)\}.$$

Determine with proof, those values of α for which f_α is differentiable.

Question 2

Find the area enclosed by the curve in the Euclidean plane: $x^{2/3} + y^{2/3} = 1$.

Question 3

Let $f : [0,1] \to \mathbb{R}$ be a function.
For $x \in [0,1]$, define $\mathrm{osc}_x(f) = \limsup_{t \to x} f(t) - \liminf_{t \to x} f(t)$.
For $0 < k \in \mathbb{R}$, let $\mathbb{D}_k = \{x \in [0,1] : \mathrm{osc}_x(f) \geq k\}$.
Prove that the set $\mathbb{D}_k$ is closed for each $k \in \mathbb{R}$.
Hence, or otherwise, show that the set of points where f is discontinuous cannot be the set of irrational real numbers.

Question 4

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be given by the formula, valid for any $x \in \mathbb{R}^n$:

$$F(x) = A(x) + B(x, x).$$

Here $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map and $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is a symmetric bilinear form on $\mathbb{R}^n$, with values in $\mathbb{R}^m$.

- Prove that A is injective if and only if F is injective near the origin.
- Prove that if A is surjective, then F is surjective near the origin.
- Prove that A is an isomorphism, if and only if F is smoothly invertible near the origin.

Question 5

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal linear transformation, so $||A(x)|| = ||x||$, for any $x \in \mathbb{R}^n$.
Let $u : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^2$ and harmonic: $\nabla.\nabla u = 0$.
Prove that the composition $u \circ A$ is also harmonic.

Question 6

Let $f : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^2$.
Suppose that $|f''(s)| \leq 1$ for all $s \in [0, 2]$.
Suppose also that the function f has a local minimum at $s = 0$.
Let $\mathbb{E}$ denote the closed unit ball, centered at the origin, in $\mathbb{R}^2$.
Show that:

$$\int_{\mathbb{E}} \int_{\mathbb{E}} \left(f(||x|| + ||y||) - f(||y||) \right) d^2x \, d^2y \leq \frac{25\pi^2}{36}.$$

Here $||(p, q)|| = \sqrt{p^2 + q^2}$, for any $(p, q) \in \mathbb{R}^2$.

Question 7

Let $f : (0, \infty) \to (0, \infty)$ be a $\mathcal{C}^2$ function, such that $f''(x) < 0$, for all $x \in (0, \infty)$.
Show that the following series $\mathcal{A}$ and $\mathcal{B}$ either both converge or both diverge:

$$\mathcal{A} = \sum_{n=1}^{\infty} f'(n), \qquad \mathcal{B} = \sum_{n=1}^{\infty} \frac{f'(n)}{f(n)}.$$

Question 8

Let $\mathcal{F} \subset C^{\infty}[0, 1]$ be a uniformly bounded and equicontinuous family of smooth functions on $[0, 1]$ such that $f' \in \mathcal{F}$ whenever $f \in \mathcal{F}$.

Suppose that $\sup_{x \in [0,1]} |f'(x) - g'(x)| \leq \frac{1}{2} \sup_{x \in [0,1]} |f(x) - g(x)|$ for all $f, g \in \mathcal{F}$.
Show that there exists a sequence f_n of functions in $\mathcal{F}$ that tends uniformly to Ce^x, for some real constant C.