Preliminary Examination in Mathematics
August 15th 2014
ID number:

Twenty points per question
The best six questions will count

Question 1
Let $\alpha \in \mathbb{R}$.
Let $f_\alpha : \mathbb{R}^2 \to \mathbb{R}$ be given by the formulas:

$$f_\alpha(0,0) = 0,$$

$$f_\alpha(x,y) = \frac{x^4 + y^4}{(x^2 + y^2)^{\alpha}}, \text{ for any } (x,y) \in \mathbb{R}^2 - \{(0,0)\}.$$

Determine with proof, those values of $\alpha$ for which $f_\alpha$ is differentiable.

Question 2
Find the area enclosed by the curve in the Euclidean plane: $x^{2/3} + y^{2/3} = 1$.

Question 3
Let $f : [0,1] \to \mathbb{R}$ be a function.
For $x \in [0,1]$, define $\text{osc}_x(f) = \limsup_{t \to x} f(t) - \liminf_{t \to x} f(t)$.
For $0 < k \in \mathbb{R}$, let $D_k = \{x \in [0,1] : \text{osc}_x(f) \geq k\}$.
Prove that the set $D_k$ is closed for each $k \in \mathbb{R}$.
Hence, or otherwise, show that the set of points where $f$ is discontinuous cannot be the set of irrational real numbers.
Question 4

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be given by the formula, valid for any $x \in \mathbb{R}^n$:

$$F(x) = A(x) + B(x, x).$$

Here $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map and $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is a symmetric bilinear form on $\mathbb{R}^n$, with values in $\mathbb{R}^m$.

- Prove that $A$ is injective if and only if $F$ is injective near the origin.
- Prove that if $A$ is surjective, then $F$ is surjective near the origin.
- Prove that $A$ is an isomorphism, if and only if $F$ is smoothly invertible near the origin.

Question 5

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal linear transformation, so $||A(x)|| = ||x||$, for any $x \in \mathbb{R}^n$.

Let $u : \mathbb{R}^n \to \mathbb{R}$ be $C^2$ and harmonic: $\nabla \cdot \nabla u = 0$.

Prove that the composition $u \circ A$ is also harmonic.

Question 6

Let $f : \mathbb{R} \to \mathbb{R}$ be $C^2$.

Suppose that $|f''(s)| \leq 1$ for all $s \in [0, 2]$.

Suppose also that the function $f$ has a local minimum at $s = 0$.

Let $E$ denote the closed unit ball, centered at the origin, in $\mathbb{R}^2$.

Show that:

$$\int_E \int_E (f(||x|| + ||y||) - f(||y||)) \, d^2x \, d^2y \leq \frac{25\pi^2}{36}.$$

Here $|| (p, q) || = \sqrt{p^2 + q^2}$, for any $(p, q) \in \mathbb{R}^2$. 

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Question 7

Let $f : (0, \infty) \rightarrow (0, \infty)$ be a $C^2$ function, such that $f''(x) < 0$, for all $x \in (0, \infty)$.
Show that the following series $\mathcal{A}$ and $\mathcal{B}$ either both converge or both diverge:

$$\mathcal{A} = \sum_{n=1}^{\infty} f'(n), \quad \mathcal{B} = \sum_{n=1}^{\infty} \frac{f'(n)}{f(n)}.$$ 

Question 8

Let $\mathcal{F} \subset C^\infty[0,1]$ be a uniformly bounded and equicontinuous family of smooth functions on $[0,1]$ such that $f' \in \mathcal{F}$ whenever $f \in \mathcal{F}$.

Suppose that $\sup_{x \in [0,1]} |f'(x) - g'(x)| \leq \frac{1}{2} \sup_{x \in [0,1]} |f(x) - g(x)|$ for all $f, g \in \mathcal{F}$.

Show that there exists a sequence $f_n$ of functions in $\mathcal{F}$ that tends uniformly to $Ce^x$, for some real constant $C$. 

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