Linear Algebra Preliminary Exam
August 2013

Problem 1 *Let A be a $n \times n$ real matrix. Show that the null space of A is the same as the null space of $A^T A$.*

Problem 2 *Suppose B is a projection and A^2 commutes with B. Is it true that A commutes with B? Prove of provide a counterexample.*

Problem 3 *Suppose*

$$\sum_{j=1}^{m} A_j = I.$$

Show that

$$\sum_{j=1}^{m} \operatorname{rank} A_j = n$$

if and only if for every $1 \leq j \leq m$ $A_j^2 = A_j$.

Problem 4 *Suppose V is the space of complex polynomials in x and y of total degree at most 3, i.e.*

$$V = \{ \sum_{i,j \geq 0, i+j \leq 3} a_{ij} x^i y^j | a_{ij} \in \mathbb{C} \}.$$

Suppose $D_x : V \to V$ is the partial derivative with respect to x. Find the Jordan canonical form of D_x.

Problem 5 *Let A, B be two $n \times n$ matrices, show that*

$$\det (I - AB) = \det (I - BA).$$

Problem 6 *Let A, B be two $n \times n$ real orthogonal matrices. Show that*

$$\det A + \det B = 0$$

implies that

$$\det (A + B) = 0.$$

Ph.D. Preliminary Examination (Analysis)

August, 2013

INSTRUCTIONS: *Do all six problems. In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should contain the necessary details. All problems are worth the same number of points.*

1. Let X be a nonempty set, and for any two functions $f, g \in \mathbb{R}^X := \{h : h \text{ is a function from } X \text{ to } \mathbb{R}\}$, let

$$d(f,g) = \sup_{x \in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$

Prove that $(\mathbb{R}^X, d)$ is a metric space.

2. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{x^n+1}$ is not uniformly convergent on $[0,1)$, but it defines a continuous function on $[0,1)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable, $a, b \in \mathbb{R}$ and $a < b$. Suppose that $f(a) = f(b) = 0$ and $|f''(x)| \leq 1$ for every $x \in [a,b]$. Prove that

$$|f(x)| \leq \frac{(b-a)^2}{8}$$

for every $x \in [a,b]$.

4. Let $f(x) = f(x_1, x_2, x_3, x_4)$ be a C^1 function from $\mathbb{R}^4$ to $\mathbb{R}$ such that $f(\mathbf{0}) = 0$, where $\mathbf{0}$ denotes the origin in $\mathbb{R}^4$. Suppose that

$$(\partial f(\mathbf{0})/\partial x_3, \partial f(\mathbf{0})/\partial x_4) \neq (0,0).$$

Prove that there exist an open neighborhood U of $(0,0)$ in $\mathbb{R}^2$ and two C^1 functions $\phi, \psi : U \to \mathbb{R}$ such that

$$f(s,t,\phi(s,t),\psi(s,t)) = 0 = f(s,t,-\psi(s,t),\phi(s,t))$$

holds for every $(s,t) \in U$.

5. Let $\Omega_1 = [0,2] \times [0,1]$ and $\Omega_2 = [1,3] \times [0,1]$. For each $j \in \{1,2\}$, let $f_j : \Omega_j \to \mathbb{R}$ be Riemann integrable over Ω_j. Define $F : [0,3] \times [0,1] \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} f_1(x,y) & \text{if } (x,y) \in \Omega_1 \backslash \Omega_2; \\ f_2(x,y) & \text{if } (x,y) \in \Omega_2 \backslash \Omega_1; \\ \inf\{f_1(x,y), f_2(x,y)\} & \text{if } (x,y) \in \Omega_1 \cap \Omega_2. \end{cases}$$

Prove that F is Riemann integrable over $[0,3] \times [0,1]$.

6. Let $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ and $f \in C^\infty(\mathbb{R}^3)$. Suppose that $f(x,y,z) = 0$ for all $(x,y,z) \in \partial\Omega$. Prove that

$$\left| \iiint_\Omega f(x,y,z)\,dV \right| \leq \frac{2\sqrt{5\pi}}{15} \left(\iiint_\Omega |\nabla f(x,y,z)|^2 dV \right)^{1/2}.$$