# EXISTENCE AND ERGODICITY FOR THE TWO-DIMENSIONAL STOCHASTIC BOUSSINESQ EQUATION

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**Abstract.** The existence of solutions to the Boussinesq system driven by random exterior forcing terms both in the velocity field and the temperature is proven using a semigroup approach. We also obtain the existence and uniqueness of an invariant measure via coupling methods.

**1. Introduction.** We study the existence and ergodicity of the stochastic Boussinesq equation

$$\begin{aligned}
&du = (\nu\Delta u - (u\cdot\nabla)u + \sigma\theta - \nabla p)dt + \sqrt{Q_1}dW_1(t),\\
&d\theta = (\chi\Delta\theta - (u\cdot\nabla)\theta)dt + \sqrt{Q_2}dW_2(t),\\
&\nabla\cdot u = 0 \quad \text{in } (0,+\infty)\times\mathcal{O},\\
&u = 0, \quad \theta = 0 \quad \text{on } (0,+\infty)\times\partial\mathcal{O},\\
&u(0,x) = u_0(x), \quad \theta(0,x) = \theta_0(x) \quad \text{in } \mathcal{O},
\end{aligned} \tag{1.1}$$

which models the interactions between an incompressible fluid flow coupled with thermal dynamics in two dimensions, in the presence of random perturbations. Here $\mathcal{O}\subset\mathbb{R}^2$ is a bounded, open and simply connected domain with smooth boundary $\partial\mathcal{O}$, and $u=(u_1,u_2)$ denotes the fluid velocity field, $\theta$ is the temperature of the fluid, $p$ stands for the pressure, $\nu$ is the kinematic viscosity and $\chi$ is the thermal diffusivity, $\sigma$ is a constant two component-vector. Also $W_1$ and $W_2$ represent two independent cylindrical Wiener processes [10, 12] defined, respectively, on a filtered space $(\Omega,\mathcal{F}_t,\mathbb{P})$ taking values in

$$H = \left\{ v\in \left(L^2(\mathcal{O})\right)^2 : \nabla\cdot v = 0 \text{ in } \mathcal{O}, \quad v\cdot n = 0 \text{ on } \partial\mathcal{O} \right\}, \quad H_1 = L^2(\mathcal{O}).$$

Finally, $Q_1$ and $Q_2$ are linear continuous, positive and symmetric operators on $H$ and $H_1$, respectively, of trace class (see Definition 5.1 in the Appendix 5), i.e., $Tr\,Q_i<\infty$, $i=1,2$, satisfying the following condition:

$$Q_1 = A^{-\gamma}, \qquad Q_2 = A_1^{-\gamma}, \tag{1.2}$$

where $1/2<\gamma<1$, $A$ and $A_1$ are as defined in (2.1).

Herein we prove the existence and uniqueness of a solution $(u(t,u_0,\theta_0),\theta(t,u_0,\theta_0))$ of the stochastic Boussinesq system (1.1), and of the corresponding invariant measure in the space $H\times H_1$. The deterministic version of the Boussinesq system (1.1) was comprehensively studied in the literature (see, e.g. [1, 9, 13] and the references therein). In the case of two-dimensional Navier-Stokes equations, the existence and uniqueness of a solution, the uniqueness of the invariant measure and properties of the corresponding Kolmogorov operators were studied in [3, 5, 4, 8, 7]. For the two-dimensional magnetohydrodynamics system, see [2].

The paper is organized as follows. In Section 2 we formulate problem (1.1) in an appropriate functional setting (see [13, 6, 12, 10]) and in Section 3 we give the main existence and uniqueness result for (1.1) which is proved via an approximating regularizing scheme. In Section 4 we prove the existence of an invariant measure $\mu$ corresponding to the stochastic flow $t\mapsto(u(t),\theta(t))$, and its uniqueness via coupling methods,

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following [11, 2]. In particular, the uniqueness of the invariant measure implies that the flow is ergodic, i.e.,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(u(t),\theta(t))\,dt = \int_{H\times H} \phi\,d\mu \qquad \forall \phi \in L^2(H\times H;\mu)$$

which agrees with some physical hypothesis on the Boussinesq flow.

**2. Functional setting and formulation of the problem.** We introduce the functional spaces to represent the coupled Navier-Stokes and heat equations (1.1) as infinite dimensional differential equation

$$V = \left\{ v \in \left(H_0^1(\mathcal{O})\right)^2 : \nabla\cdot v = 0 \text{ in } \mathcal{O} \right\}, \quad V_1 = H_0^1(\mathcal{O}).$$

The norms of $V$ and $V_1$ are denoted by the same symbol $\|\cdot\|$:

$$\begin{aligned} \|v\|^2 &= \sum_{i=1}^2 \int_{\mathcal{O}} |\nabla v_i|^2 dx, \quad v=(v_1,v_2)\in V,\\ \|\eta\|^2 &= \int_{\mathcal{O}} |\nabla \eta|^2 dx, \quad \eta \in V_1. \end{aligned}$$

Let denote by $V'$ and $V_1' = H^{-1}(\mathcal{O})$ the duals of $V$ and $V_1$ respectively, endowed with the dual norms. Denote again $(\cdot,\cdot)$ the scalar product on $H$ and the pairing between $V$ and $V'$, $V_1$ and $V_1'$. The norm on $H$ and $L^2(\Omega)$ will both be denoted by $|\cdot|$. Identifying $H$ with its own dual we have $V\subset H\subset V'$. Let $A\in L(V,V'), A_1 \in L(V_1,V_1'), b : V\times V\times V \to \mathbb{R}$ be defined by

$$\begin{aligned} (Av,w) &= \int_{\mathcal{O}} \nabla v\cdot\nabla w\,dx, \quad v,w\in V,\\ (A_1\alpha,\beta) &= \int_{\mathcal{O}} \nabla\alpha\cdot\nabla\beta dx, \quad \alpha,\beta\in V_1,\\ b(u,v,w) &= \sum_{i,j=1}^2 \int_{\mathcal{O}} u_i D_i v_j w_j dx, \quad u,v,w\in V, \end{aligned} \tag{2.1}$$

and $B : V\to V'$ given by

$$(Bv,w) = b(v,v,w), \quad v,w\in V.$$

Then system (1.1) can be written as

$$\begin{aligned} &du + (\nu Au + B(u) - \sigma\theta)dt = \sqrt{Q_1}dW_1(t),\\ &d\theta + (\chi A_1\theta + (u\cdot\nabla)\theta)dt = \sqrt{Q_2}dW_2(t),\\ &u(0)=u_0, \quad \theta(0)=\theta_0. \end{aligned} \tag{2.2}$$

The cylindrical Wiener process $W=(W_1,W_2)$ on $H\times H$ is defined [10] by

$$W_i(t) = \sum_{j=1}^\infty \beta_j^i(t)e_j^i, \qquad i=1,2,$$

where $\{e_j^1\}, \{e_j^i\}$ are two complete orthonormal bases of eigenfunctions of $A$, respectively $A_1$, and $\{\beta_j^i\}, i = 1,2$ are two independent sequences of mutually independent Brownian motions on the filtered space

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We denote by $C_W(0,T;H\times H_1)$ the space of all continuous functions $Z:[0,T]\in L^2(\Omega, \mathcal{F}_t, \mathbb{P};H\times H_1)$ which are adapted to the filtration $\mathcal{F}_t$. The spaces $L^2_W(0,T;V\times V)$ and $L^2_W(0,T;V'\times V_1')$ are similarly defined.

Consider the stochastic convolution that is the mild solution of the problem

$$\begin{aligned} &dW_{\mathcal{A}}(t) + \mathcal{A}W_{\mathcal{A}}(t)dt = \sqrt{Q}dW(t),\\ &W_{\mathcal{A}}(0) = 0, \end{aligned} \tag{2.3}$$

given by

$$W_{\mathcal{A}}(t) = \int_0^t e^{-\mathcal{A}(t-s)}\sqrt{Q}dW(s) := (W_A^1(t), W_A^2(t)),$$

where

$$A = \begin{pmatrix} \nu A & 0 \\ 0 & \chi A_1 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Under our assumptions it follows that [4]

$$W_{\mathcal{A}} \in C_W(0,T;H\times H) \cap (L^4_W([0,T]\times\mathcal{O}))^2,$$

and by Theorem 2.13 of [4] we have that

$$\mathbb{E}\left(\sup_{(t,x)\in[0,T]\times\mathcal{O}} |W_A^i|^4\right) < +\infty.$$

**3. Existence and uniqueness result.** Our main theorem is as follows.

THEOREM 3.1. *For all $(u_0,\theta_0)\in H\times H_1$ and $T>0$ problem (2.2) has a unique solution $(u,\theta)\in L^2_W(0,T;V\times V_1)$.*

To prove Theorem 3.1 we reduce (2.2) to a deterministic equation with random coefficients, via the substitution

$$u(t) = v(t) + W_A^1(t), \quad \theta(t) = \eta(t) + W_A^2(t).$$

Then (2.2) reduces to

$$\begin{aligned} &v' + \nu Av + B(v) + v\cdot\nabla W_A^1 + W_A^1\cdot\nabla v - \sigma\theta - \sigma W_A^2 = -B(W_A^1),\\ &\eta' + \chi A_1\eta + v\cdot\nabla\eta + v\cdot\nabla W_A^2 + W_A^1\cdot\nabla\eta = -W_A^1\cdot\nabla W_A^2,\\ &v(0) = u_0, \quad \eta(0) = \theta_0. \end{aligned} \tag{3.1}$$

We recall the following standard estimates, which will be used in the sequel:

$$\begin{aligned} &|(B(v),w)| \le C|v|\|v\|\|w\| \quad\Rightarrow\quad \|B(v)\|_{V'} \le C|v|\|v\|,\\ &|(v\cdot\nabla\eta,\alpha)| \le C|v|^{1/2}\|v\|^{1/2}|\eta|^{1/2}\|\eta\|^{1/2}\|\alpha\| \quad\Rightarrow\quad \|v\cdot\nabla\eta\|_{V_1'} \le C|v|^{1/2}\|v\|^{1/2}|\eta|^{1/2}\|\eta\|^{1/2},\\ &\|W_A^1\cdot\nabla v\|_{V'} + \|v\cdot\nabla W_A^1\|_{V'} \le C|W_A^1|^4|v|^{1/2}\|v\|^{1/2},\\ &\|v\cdot\nabla W_A^2\|_{V_1'} \le C|W_A^2|^4|v|^{1/2}\|v\|^{1/2},\\ &\|W_A^2\cdot\nabla\eta\|_{V_1'} \le C|W_A^1|^4|\eta|^{1/2}\|\eta\|^{1/2}. \end{aligned}$$

PROPOSITION 3.2. *Let $(u_0,\theta_0)\in H\times H_1$. Then there is a unique solution $(v,\eta)\in L^2_W(0,T;V\times V_1)$ to (3.1) such that $\mathbb{P}$-a.s. $(v,\eta):[0,T]\to V'\times V_1'$ is absolutely continuous on $[0,T]$ and*

*(i)* $v' \in L^2(0,T;V')$, $\eta' \in L^2(0,T;V_1')$, $\mathbb{P}$*-a.s.*
*(ii)* $v \in C([0,T],H)$ *and* $\eta \in C([0,T],H_1)$, $\mathbb{P}$*-a.s.*

*Proof.* We consider the approximating equation

$$\begin{aligned}
&v_\varepsilon' + \nu A v_\varepsilon + \Phi_1^\varepsilon(v_\varepsilon) + v_\varepsilon \cdot \nabla W_A^1 + W_A^1 \cdot \nabla v_\varepsilon - \sigma\theta_\varepsilon - \sigma W_A^2 = -B(W_A^1),\\
&\eta_\varepsilon' + \chi A_1 \eta_\varepsilon + \Phi_2^\varepsilon(v_\varepsilon, \eta_\varepsilon) + v_\varepsilon \cdot \nabla W_A^2 + W_A^1 \cdot \nabla \eta_\varepsilon = -W_A^1 \cdot \nabla W_A^2,\\
&v(0) = u_0, \quad \eta(0) = \theta_0,
\end{aligned} \tag{3.2}$$

where

$$\Phi_1^\varepsilon(v_\varepsilon) = \begin{cases} B(v) & \text{if } \|v\| \le \frac{1}{\varepsilon},\\ \frac{B(v)}{\varepsilon^2\|v\|^2} & \text{if } \|v\| > \frac{1}{\varepsilon}. \end{cases}$$

and

$$\Phi_2^\varepsilon(v_\varepsilon, \theta_\varepsilon) = \begin{cases} v \cdot \nabla \eta & \text{if } \|v\| + \|\eta\| \le \frac{1}{\varepsilon},\\ \frac{v \cdot \nabla \eta}{\varepsilon^2(\|v\| + \|\eta\|)^2} & \text{if } \|v\| + \|\eta\| > \frac{1}{\varepsilon}. \end{cases}$$

It is easy to see that $u_\varepsilon = v_\varepsilon + W_A^1$ and $\theta_\varepsilon = \eta_\varepsilon + W_A^2$ satisfy

$$\begin{aligned}
&du_\varepsilon + (\nu A u_\varepsilon + \Phi_1^\varepsilon - \sigma\theta_\varepsilon)dt = \sqrt{Q_1}dW_1(t),\\
&d\theta_\varepsilon + (\chi A_1 \theta_\varepsilon + \Phi_2^\varepsilon)dt = \sqrt{Q_2}dW_2(t),\\
&u(0) = u_0, \quad \theta(0) = \theta_0.
\end{aligned} \tag{3.3}$$

Multiplying the first and second equations of (3.2) by $v_\varepsilon$ and $\theta_\varepsilon$ respectively, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|v_\varepsilon|^2 + |\eta_\varepsilon|^2\right) + \nu\|v_\varepsilon\|^2 + \chi\|\eta_\varepsilon\|^2 + b(v_\varepsilon, W_A^1, v_\varepsilon) + b(v_\varepsilon, W_A^2, \eta_\varepsilon)\\
&= (\sigma\eta_\varepsilon, v_\varepsilon) + (\sigma W_A^2, v_\varepsilon) - b(W_A^1, W_A^1, v_\varepsilon) - b(W_A^1, W_A^2, \eta_\varepsilon).
\end{aligned}$$

Recall Young's inequality: $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ for $p$ and $q$ conjugate. Then we have

$$\begin{aligned}
&b(v_\varepsilon, W_A^1, v_\varepsilon) \le C|v_\varepsilon|^{1/2}\|v_\varepsilon\|^{3/2}|W_A^1|_4 \le \frac{\nu}{3}\|v_\varepsilon\|^2 + C|v_\varepsilon|^2|W_A^1|_4^4,\\
&b(v_\varepsilon, W_A^2, \eta_\varepsilon) \le C|v_\varepsilon|^{1/2}\|v_\varepsilon\|^{1/2}|\|\eta_\varepsilon\||W_A^2|_4\\
&\quad \le C|v_\varepsilon|\|v_\varepsilon\||W_A^2|_4^2 + \frac{\chi}{2}\|\eta_\varepsilon\|^2\\
&\quad \le \frac{\nu}{3}\|v_\varepsilon\|^2 + C|v_\varepsilon|^2|W_A^2|_4^4 + \frac{\chi}{2}\|\eta_\varepsilon\|^2,\\
&b(W_A^1, W_A^1, v_\varepsilon) \le C|W_A^1|_4^2\|v_\varepsilon\| \le \frac{\nu}{3}\|v_\varepsilon\|^2 + C|W_A^1|_4^4,\\
&b(W_A^1, W_A^2, \eta_\varepsilon) \le C|W_A^1|_4|W_A^2|_4\|\eta_\varepsilon\| \le \frac{\chi}{2}\|\eta_\varepsilon\|^2 + C|W_A^1|_4^2|W_A^2|_4^2\\
&(\sigma\eta_\varepsilon, v_\varepsilon) \le C(|\eta_\varepsilon|^2 + |v_\varepsilon|^2).
\end{aligned}$$

So we have

$$\frac{d}{dt}\left(|v_\varepsilon|^2 + |\eta_\varepsilon|^2\right) + \nu\|v_\varepsilon\|^2 + \chi\|\eta_\varepsilon\|^2 \le C(|W_A^1|_4^4 + |W_A^2|_4^4 + C)(|\eta_\varepsilon|^2 + |v_\varepsilon|^2 + 1). \tag{3.4}$$

Integrating (3.4) with respect to $t \in [0,T]$ and using Gronwall's inequality, we have

$$
\begin{aligned}
&|v_\varepsilon(t)|^2 + |\eta_\varepsilon(t)|^2 + \int_0^T (\|v_\varepsilon(s)\|^2 + \|\eta_\varepsilon(s)\|^2) ds \\
&\quad \le C(|u_0|^2 + |\theta_0|^2) \exp\left( C \int_0^T (|W_A^1|_4^4 + |W_A^2|_4^4 + C) ds \right) + C, \quad t \in [0,T],
\end{aligned} \tag{3.5}
$$

where $C$ is independent of $\varepsilon$ and $\omega$.

Now we fix $\omega \in \Omega$ and select a sub-sequence $\varepsilon = \varepsilon(\omega)$ such that

$$
\begin{aligned}
&v_\varepsilon(t) \to v(t) \quad \text{weakly in } L^2(0,T;V), \text{ weak star in } L^\infty(0,T;H), \\
&\eta_\varepsilon(t) \to \eta(t) \quad \text{weakly in } L^2(0,T;V_1), \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \\
&Av_\varepsilon(t) \to Av(t) \quad \text{weakly in } L^2(0,T;V'), \\
&A\eta_\varepsilon(t) \to A\eta(t) \quad \text{weakly in } L^2(0,T;V_1'),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\Phi_1^\varepsilon(v_\varepsilon(t)) \to \varphi_1(t) \quad \text{weakly in } L^2(0,T;V') \\
&\Phi_2^\varepsilon(v_\varepsilon(t), \eta_\varepsilon(t)) \to \varphi_2(t) \quad \text{weakly in } L^2(0,T;V_1') \\
&v_\varepsilon(t) \cdot \nabla W_A^1 \to v(t) \cdot W_A^1 \quad \text{weakly in } L^2(0,T;V') \\
&W_A^1 \cdot \nabla v_\varepsilon(t) \to W_A^1 \cdot \nabla v(t) \quad \text{weakly in } L^2(0,T;V') \\
&\sigma\eta_\varepsilon(t) \to \sigma\eta(t) \quad \text{weakly in } L^2(0,T;V_1') \\
&v_\varepsilon(t) \cdot \nabla W_A^2 \to v(t) \cdot W_A^2 \quad \text{weakly in } L^2(0,T;V_1') \\
&W_A^1 \cdot \nabla \eta_\varepsilon(t) \to W_A^1 \cdot \nabla \eta(t) \quad \text{weakly in } L^2(0,T;V_1').
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&v' + \nu A v + \varphi_1 + v \cdot \nabla W_A^1 + W_A^1 \cdot \nabla v = -B(W_A^1) + \sigma\theta + \sigma W_A^2, \text{ a.e. } t \in [0,T], \\
&\eta' + \chi A_1 \eta + \varphi_2 + v \cdot \nabla W_A^2 + W_A^1 \cdot \nabla \eta = -W_A^1 \cdot \nabla W_A^2, \text{ a.e. } t \in [0,T], \\
&v(0) = u_0, \quad \eta(0) = \theta_0.
\end{aligned} \tag{3.6}
$$

On the other hand, since $v_\varepsilon'$ and $\eta_\varepsilon'$ are bounded in $L^2(0,T;V')$ and $L^2(0,T;V_1')$ respectively, we also have that for $\varepsilon \to 0$

$$
\begin{aligned}
&v_\varepsilon \to v \text{ strongly in } L^2(0,T;H), \\
&\eta_\varepsilon \to \eta \text{ strongly in } L^2(0,T;L^2(\mathcal{O})).
\end{aligned}
$$

Moreover,

$$
\int_0^T (\varphi_1(t), \psi(t)) dt \to \int_0^T b(v,v,\psi) dt, \quad \forall \psi \in C([0,T], D(A)), \tag{3.7}
$$

and the reason is as follows.

$$
\begin{aligned}
&\int_0^T (\varphi_1(t), \psi(t)) dt \\
&= \int_{t\in[0,T]:\|v_\varepsilon\| \le 1/\varepsilon} b(v_\varepsilon, v_\varepsilon, \psi) dt + \int_{t\in[0,T]:\|v_\varepsilon\| > 1/\varepsilon} \frac{b(v_\varepsilon, v_\varepsilon, \psi)}{\varepsilon^2 \|v_\varepsilon^2\|} dt \\
&= I_\varepsilon^1 + I_\varepsilon^2.
\end{aligned}
$$

We have shown that

$$b(v_\varepsilon, v_\varepsilon, \psi) \to b(v, v, \psi), \quad \text{a.e. } t \in [0, T].$$

Since

$$|b(v_\varepsilon, v_\varepsilon, \psi)| \le C|v_\varepsilon|\|v_\varepsilon\|,$$

we infer by the dominated convergence theorem that

$$I_\varepsilon^1 \to \int_0^T b(v, v, \psi)dt \quad \text{as } \varepsilon \to 0.$$

On the other hand, we have shown that

$$\sup_{t\in[0,T]} \{\|v_\varepsilon(t)\| > 1/\varepsilon\} \le C\varepsilon^2.$$

Therefore,

$$|I_\varepsilon^2| \le C\varepsilon^2 \frac{|v_\varepsilon|\|v_\varepsilon\|\|\psi\|}{\varepsilon^2\|v_\varepsilon\|^2} \le C\frac{1}{\|v_\varepsilon\|} \le C\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0.$$

Thus, it follows that $\varphi_1(t) = B(v(t))$, a.e. $t \in [0, T]$. Similarly, we have $\varphi_2(t) = v \cdot \nabla\eta$.

This means that the pair $(v, \eta)$ is a solution to (3.1) for every fixed $\omega \in \Omega$. On the other hand, it is readily seen that for each $\omega \in \Omega$, (3.6) with $\varphi_1 = B(v)$ and $\varphi_2 = v \cdot \nabla\eta$ has at most one solution $(v, \eta)$ with the above properties. This implies that, for $\varepsilon \to 0$,

$$v_\varepsilon(t) \to v(t), \quad \eta_\varepsilon(t) \to \eta(t),$$

weakly in $L^2(0, T; V)$ and $L^2(0, T; V_1)$, respectively, $\mathbb{P}$-a.s. This indicates that $v$ and $\eta$ (and $v'$ and $\eta'$) are adapted to the filtration $\mathcal{F}_t$ and therefore $(v, \eta) \in L^2_W(0, T; V \times V_1)$ and $(v', \eta') \in L^2_W(0, T; V' \times V_1')$. □

Now we are ready to prove Theorem 3.1.

*Proof.* [Proof of Theorem 3.1] For the first equation of (3.3), we have by Ito's formula

$$\frac{1}{2}\mathbb{E}|u_\varepsilon(t)| + \nu\mathbb{E}\int_0^t \|u_\varepsilon(s)\|^2 ds = \frac{1}{2}|u_0|^2 + \frac{1}{2}t\,Tr\,Q_1 + \mathbb{E}\int_0^t (\sigma\theta_\varepsilon, u_\varepsilon)ds. \tag{3.8}$$

Proceeding similarly as in the second equation in (3.3), we obtain

$$\frac{1}{2}\mathbb{E}|\theta_\varepsilon(t)| + \chi\mathbb{E}\int_0^t \|\theta_\varepsilon(s)\|^2 ds = \frac{1}{2}|\theta_0|^2 + \frac{1}{2}t\,Tr\,Q_2. \tag{3.9}$$

Combining (3.8) and (3.9) we get, for $t \in [0, T]$

$$\begin{aligned} &\mathbb{E}(|u_\varepsilon|^2 + |\theta_\varepsilon|^2) + 2\mathbb{E}\int_0^t (\nu\|u_\varepsilon(s)\|^2 + \chi\|\theta_\varepsilon(s)\|^2)ds \\ &\quad = |u_0|^2 + |\theta_0|^2 + t\,Tr\,(Q_1 + Q_2) + 2\mathbb{E}\int_0^t (\sigma\theta_\varepsilon, u_\varepsilon)ds. \end{aligned} \tag{3.10}$$

By Gronwall's inequality, we deduce from (3.10) that

$$\mathbb{E}(|u_\varepsilon|^2 + |\theta_\varepsilon|^2) + \mathbb{E}\int_0^t (\|u_\varepsilon(s)\|^2 + \|\theta_\varepsilon(s)\|^2)ds \le C. \tag{3.11}$$

This implies that, for $\varepsilon \to 0$,

$$\begin{aligned} u_\varepsilon \to u = v + W_A^1 \quad & \text{weakly in } L^2_W(0,T;V), \\ \theta_\varepsilon \to \theta = \eta + W_A^2 \quad & \text{weakly in } L^2_W(0,T;V_1), \end{aligned}$$

where $(u,\theta)$ is a solution to (1.1).

As for uniqueness, if $(\tilde{u}(t), \tilde{\theta}(t))$ is a solution with initial data $(u_1, \theta_1)$ we have by (2.2) that

$$\begin{aligned} &\frac{1}{2} d(|u(t)-\tilde{u}(t)|^2 + |\theta(t)-\tilde{\theta}(t)|^2) + \nu\|u(t)-\tilde{u}(t)\|^2 + \chi\|\theta(t)-\tilde{\theta}(t)\|^2 \\ &\le |b(u-\tilde{u},\tilde{u},u-\tilde{u})| + |((u-\tilde{u})\cdot\nabla\tilde{\theta},\theta-\tilde{\theta})| + |(\sigma(\theta-\tilde{\theta}),u-\tilde{u})| \\ &\le C|u-\tilde{u}|\|u-\tilde{u}\|\|\tilde{u}\| + C|u-\tilde{u}|^{1/2}\|u-\tilde{u}\|^{1/2}|\tilde{\theta}|^{1/2}\|\tilde{\theta}\|^{1/2}\|\theta-\tilde{\theta}\| + C|\theta-\tilde{\theta}||u-\tilde{u}| \\ &\le C|u-\tilde{u}|^2\|\tilde{u}\|^2 + \frac{\nu}{4}\|u-\tilde{u}\|^2 + C|u-\tilde{u}|^2|\tilde{\theta}|^2\|\tilde{\theta}\|^2 + \frac{\nu}{4}\|u-\tilde{u}\|^2 + \frac{\chi}{2}\|\theta-\tilde{\theta}\|^2 + C(|\theta-\tilde{\theta}|^2 + |u-\tilde{u}|^2) \\ &\le C(|\theta-\tilde{\theta}|^2 + |u-\tilde{u}|^2)(1+\|\tilde{u}\|^2 + |\tilde{\theta}|^2\|\tilde{\theta}\|^2) + \frac{\nu}{2}\|u-\tilde{u}\|^2 + \frac{\chi}{2}\|\theta-\tilde{\theta}\|^2. \end{aligned}$$

Using Gronwall's inequality there holds

$$\begin{aligned} &|u(t)-\tilde{u}(t)|^2 + |\theta(t)-\tilde{\theta}(t)|^2 \\ &\le C(|u_0-u_1|^2 + |\theta_0-\theta_1|^2) \times \exp\left( C\int_0^t (1+\|\tilde{u}\|^2 + |\tilde{\theta}|^2\|\tilde{\theta}\|^2) ds \right). \end{aligned}$$

This completes the uniqueness of $(u,\theta)$ as well as the continuity of $(u_0,\theta_0) \to (u(t),\theta(t))$. □

## 4. Ergodicity.

**4.1. Existence of invariant measure.** Let $(u(t,u_0,\theta_0), \theta(t,u_0,\theta_0)) \in L^2_W(0,T;V\times V_1)$ be the solution of (1.1) with initial data $(u_0,\theta_0)$. Set

$$P_t\phi(u_0,\theta_0) = \mathbb{E}[\phi(u(t,u_0,\theta_0),\theta(t,u_0,\theta_0))], \quad \forall (u_0,\theta_0)\in H\times H_1,\ \phi\in C_b(H\times H_1).$$

Recall that a Borel probability measure $\mu$ in $H\times H_1$ is invariant (Definition 5.3) for the transition semigroup $P_t$ if

$$\int_{H\times H_1} P_t\phi d\mu = \int_{H\times H_1} \phi d\mu, \quad \forall \phi \in C_b(H\times H_1).$$

THEOREM 4.1. *There exists at least one invariant measure $\mu$ for $P_t$.*

*Proof.* From (3.10) we have that

$$\begin{aligned} &\mathbb{E}(|u(t)|^2 + |\theta(t)|^2) + \mathbb{E}\int_0^t (\|u(s)\|^2 + \|\theta(s)\|^2) ds \\ &\le C\left(|u_0|^2 + |\theta_0|^2 + tTr(Q_1+Q_2)\right), \quad t\ge 0. \end{aligned} \tag{4.1}$$

Let $\pi_t(u_0,\theta_0,\cdot)$ be the law of process $(u(t),\theta(t))$. Then

$$P_t\phi(u_0,\theta_0) = \int_0^t \phi(u_1,\theta_1)\pi_t(u_0,\theta_0,du_1,d\theta_1).$$

In order to prove the existence of an invariant measure, it is enough to show that the set

$$\mu_T := \frac{1}{T}\int_0^T \pi_t(u_0,\theta_0,\cdot)dt, \quad T>1,$$

is tight in $\mathcal{P}(H \times H_1)$ (see the definition 5.4 in the Appendix 5). With fixed $(u_0, \theta_0) \in H \times H_1$, we have that

$$\frac{1}{t}\mathbb{E}\int_0^t (\|u\|^2 + \|\theta\|^2)ds \le C(|u_0|^2 + |\theta_0|^2 + Tr(Q)).$$

Let $B_R$ denote the ball of radius $R$ in $V \times V_1$. Then $\forall R > 0$, we have

$$\begin{aligned}\mu_T(B_R^c) &= \frac{1}{T}\int_0^T \pi_t(u_0, \theta_0, B_R^c)dt \\ &\le \frac{1}{TR^2}\int_0^T \mathbb{E}(\|u\|^2 + \|\theta\|^2)ds \\ &\le \frac{1}{R^2}C(|u_0|^2 + |\theta_0|^2 + Tr(Q)),\end{aligned}$$

which yields that $\{\mu_T\}_{T\ge 1}$ is tight. □

**4.2. Uniqueness of invariant measure.** In this section we prove the uniqueness of the invariant measure $\mu$ using coupling method (see, e.g., [2, 11, 7, 8]). We follow the approach presented in [2, 11], and Lemmas 4.2-4.4 are the main steps in the proof. With these a priori estimates, the main result, Theorem 4.5, follows exactly the same framework as in [2]. Therefore, we only prove Lemmas 4.2-4.4 in this section. For a detailed proof of Theorem 4.5, please refer to [2].

Lemma 4.2. *The following estimate holds:*

$$\nu^*\mathbb{E}\int_0^t (\|u\|^2 + \|\theta\|^2)ds \le |u_0|^2 + |\theta_0|^2 + \frac{t}{2}Tr(Q), \tag{4.2}$$

*where* $\nu^* = \min\{\nu, \chi\}$.

*Proof.* This is a direct consequence of (4.1). □

Lemma 4.3. *Let* $\rho_0, \rho_1 > 0$. *Then there exist* $\alpha = \alpha(\rho_0, \rho_1)$ *and* $T = T(\rho_0, \rho_1) > 0$ *such that for any* $t \in [T, 2T]$, $|u_0| \le \rho_0, |\theta_0| \le \rho_0$, *we have*

$$\mathbb{P}(|u| \le \rho_1, |\theta| \le \rho_1) \ge \alpha. \tag{4.3}$$

*Proof.* Let $v = u - W_A^1$, $\eta = \theta - W_A^2$, where $W_A^1$ and $W_A^2$ are mild solutions to (2.3). Multiplying the second equation (3.1) with $\eta$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\eta|^2 + \chi\|\eta\|^2 &\le |b(v, W_A^2, \eta)| + |b(W_A^1, W_A^2, \eta) \\ &\le C(\|v\||W_A^2|_4 + |W_A^1|_4|W_A^2|_4) + \frac{\chi}{2}\|\eta\|^2.\end{aligned}$$

Thus,

$$\frac{d}{dt}|\eta|^2 + \chi\|\eta\|^2 \le C|W_A^2|_4(\|v\| + |W_A^1|_4),$$

equivalently,

$$\frac{d}{dt}\left(e^{\delta t}|\eta|^2\right) \le C|W_A^2|_4(\|v\| + |W_A^1|_4)e^{\delta t}. \tag{4.4}$$

Note that $W_A^1$ and $W_A^2$ are independent Gaussian processes in $L^4(\mathcal{O})$, and following the argument in [4] we have

$$\mathbb{P}\left(|W_A^1|_4^2 + |W_A^2|_4^2\right) \le \epsilon, \quad \forall t \in [0,\, 2T] > 0.$$

Integrating and rearranging (4.4) yields

$$
\begin{aligned}
|\eta(t)|^2 &\le e^{-\delta t}|\eta(0)|^2 + Ce^{-\delta t}\epsilon\int_0^t e^{\delta s}(\|v\|+\epsilon)ds \\
&\le e^{-\delta t}|\eta(0)|^2 + Ce^{-\delta t}\epsilon\left(\left(\int_0^t e^{2\delta s}ds\right)^{1/2}\left(\int_0^t \|v\|^2 ds\right)^{1/2} + \epsilon\int_0^t e^{\delta s}ds\right) \\
&\le e^{-\delta t}|\eta(0)|^2 + C\epsilon, \qquad (4.5)
\end{aligned}
$$

where we used the a priori result from (3.5). Now multiply equation (3.1) with $v$ and $\eta$ respectively, then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|v^2| + |\eta|^2) + \nu\|v\|^2 + \chi\|\eta\|^2 \\
\le& C\left((|W_A^1|_4^4 + |W_A^2|_4^4)|v|^2 + |W_A^1|_4^4 + |W_A^2|_4^4\right) + |(\sigma\eta, v)| + \frac{\nu}{4}\|v\|^2 + \frac{\chi}{2}\|\eta\|^2 \\
\le& C(|W_A^1|_4^4 + |W_A^2|_4^4)(|v|^2+1) + \frac{\nu}{4}\|v\|^2 + C_\nu\|\eta\|^2 + \frac{\nu}{4}\|v\|^2 + \frac{\chi}{2}\|\eta\|^2.
\end{aligned}
$$

Applying the estimate of (4.5) to the above equation yields

$$
\frac{d}{dt}(|v^2| + |\eta|^2) + \alpha(\|v\|^2 + \|\eta\|^2) \le C(|\eta(0)|^2 e^{-\delta t} + \epsilon). \qquad (4.6)
$$

Multiply $e^{\alpha t}$ to both sides of (4.6) and integrate from 0 to $t$, then we have

$$
|v(t)|^2 + |\eta(t)|^2 \le e^{-\alpha t}(|v(0)|^2 + |\eta(0)|^2) + C|\eta(0)|^2 e^{-\min(\alpha,\delta)t} + C\epsilon t. \qquad (4.7)
$$

The right-hand side will be small by choosing $T$ large enough first, and then letting $\epsilon$ small enough. ☐

LEMMA 4.4. *Let $g \in C_b(H\times H_1)$ be such that $\|g\|_0 \le 1$. For notational simplicity, denote $(x,y) \in H\times H_1$ by the initial values of $u$ and $\theta$. Then for any $t>0$ and $\delta>0$ such that*

$$
|P_t g(x,y) - P_t g(x_1,y_1)| \le \frac{1}{2},
$$

*for all $(x,y),(x_1,y_1) \in H\times H_1$, $x,y,x_1,y_1 \in B_\delta(0)$, where $B_\delta(0)$ denotes a disk centered at the origin with radius $\delta$.*

*Proof.* Let $Z=(u,\theta)$ be the solution of (1.1) with initial value $(x,y)\in H\times H_1$ and by $DZ$ the Gateaux derivative of $Z$. Denote

$$
\xi_1 = D_x u, \quad \xi_2 = D_x\theta, \quad \xi_3 = D_y u, \quad \xi_4 = D_y\theta,
$$

where $D_x$ and $D_y$ are Gateaux derivatives with respect to $x$ and $y$. Then

$$
\begin{aligned}
&\xi_1' + \nu A\xi_1 + B'(u)\xi_1 - \sigma\xi_2 = 0, \\
&\xi_2' + \chi A_1\xi_2 + F(u,\theta)_u\xi_1 + F(u,\theta)_\theta\xi_2 = 0, \qquad (4.8)\\
&\xi_1(0) = 1, \qquad \xi_2(0) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\xi_3' + \nu A\xi_3 + B'(u)\xi_3 - \sigma\xi_4 = 0, \\
&\xi_4' + \chi A_1\xi_4 + F(u,\theta)_u\xi_3 + F(u,\theta)_\theta\xi_4 = 0, \qquad (4.9)\\
&\xi_3(0) = 0, \qquad \xi_4(1) = 1
\end{aligned}
$$

where $(F(u,\theta),w)=b(u,\theta,w)$ for any $w\in V$. By multiplying (4.8) by $\xi_1$ and $\xi_2$, respectively, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\xi_1|^2+|\xi_2|^2)+\nu\|\xi_1\|^2+\chi\|\xi_2\|^2\\
=&-b(\xi_1,u,\xi_1)+(\sigma\xi_2,\xi_1)-b(\xi_1,\theta,\xi_2)\\
\le&C|\xi_1|\|\xi_1\|\|u\|+C(|\xi_2|^2+|\xi_1|^2)+C\|\xi_1\|^{\frac{1}{2}}\|\xi_2\|^{\frac{1}{2}}|\xi_1|^{\frac{1}{2}}|\xi_2|^{\frac{1}{2}}\|\theta\|\\
\le&\epsilon\|\xi_1\|^2+C|\xi_1|^2\|u\|^2+C(|\xi_2|^2+|\xi_1|^2)+\\
&\qquad\qquad\frac{\epsilon}{2}(\|\xi_1\|^2+\|\xi_2\|^2)+C(|\xi_1|^2+|\xi_2|^2)\|\theta\|^2.
\end{aligned}$$

For properly chosen $\epsilon$, there exists $\gamma>0$ such that

$$\begin{aligned}
&\frac{d}{dt}(|\xi_1|^2+|\xi_2|^2)+\gamma(\|\xi_1\|^2+\|\xi_2\|^2)\\
\le&C(|\xi_1|^2+|\xi_2|^2)(\|u\|^2+\|\theta\|^2+C),
\end{aligned}$$

and by Gronwall's inequality, we obtain

$$|\xi_1|^2+|\xi_2|^2+\gamma\int_0^t(\|\xi_1\|^2+\|\xi_2\|^2)ds\le C\exp\left(C\int_0^t(\|u\|^2+\|\theta\|^2+C)ds\right). \tag{4.10}$$

Similarly,

$$|\xi_3|^2+|\xi_4|^2+\gamma\int_0^t(\|\xi_3\|^2+\|\xi_4\|^2)ds\le C\exp\left(C\int_0^t(\|u\|^2+\|\theta\|^2+C)ds\right). \tag{4.11}$$

The next step is to estimate

$$\mathbb{E}\left[g\left(u(t,x,y),\theta(t,x,y)\right)-g\left(u(t,x_1,y_1),\theta(t,x_1,y_1)\right)\right]$$

by following the argument as in [11]. To do that, we introduce a cut-off function

$$\Phi_K(r)=\begin{cases}1 & \text{if } r\in[0,K]\\ 0 & \text{if } r\in[2K,\infty]\\ \in[0,1] & \text{if } r\in[K,2K].\end{cases}$$

Then

$$\begin{aligned}
&\mathbb{E}\left[g\left(u(t,x,y),\theta(t,x,y)\right)-g\left(u(t,x_1,y_1),\theta(t,x_1,y_1)\right)\right]\\
=&\mathbb{E}\left[g\left(u(t,x,y),\theta(t,x,y)\right)\times\Phi_K\left(\int_0^t(\|u(s,x,y)\|^2+\|\theta(s,x,y)\|^2)ds\right)\right]\\
&-\mathbb{E}\left[g\left(u(t,x_1,y_1),\theta(t,x_1,y_1)\right)\times\Phi_K\left(\int_0^t(\|u(s,x_1,y_1)\|^2+\|\theta(s,x_1,y_1)\|^2)ds\right)\right]\\
&+\mathbb{E}\left[g(u(t,x,y),\theta(t,x,y))\times\left(1-\Phi_K\left(\int_0^t(\|u(s,x,y)\|^2+\|\theta(s,x,y)\|^2)ds\right)\right)\right]\\
&-\mathbb{E}\left[g(u(t,x_1,y_1),\theta(t,x_1,y_1))\times\left(1-\Phi_K\left(\int_0^t(\|u(s,x_1,y_1)\|^2+\|\theta(s,x_1,y_1)\|^2)ds\right)\right)\right]\\
=&H_1(t)+H_2(t)+H_3(t).
\end{aligned}$$

As a result of Lemma 4.2, we have

$$
\begin{aligned}
|H_2(t)| &\le \mathbb{P}\left(\int_0^t (\|u(s,x,y)\|^2 + \|\theta(s,x,y)\|^2)ds \ge K\right)\|g\|_0 \\
&\le \|g\|_0 \left(\frac{|x|^2+|y|^2}{\nu^* K} + \frac{Tr(Q_1+Q_2)t}{2K}\right).
\end{aligned}
\tag{4.12}
$$

Similarly,

$$
|H_3(t)| \le \|g\|_0 \left(\frac{|x_1|^2+|y_1|^2}{\nu^* K} + \frac{Tr(Q_1+Q_2)t}{2K}\right). \tag{4.13}
$$

In order to estimate $H_1(t)$, we write it as follows.

$$
\begin{aligned}
H_1(t) = \int_0^1 \frac{d}{d\lambda}\mathbb{E}\bigg[ & g\left(u(t,x_\lambda,y_\lambda),\theta(t,x_\lambda,y_\lambda)\right) \\
& \times \Phi_K\left(\int_0^t (\|u(s,x_\lambda,y_\lambda)\|^2 + \|\theta(s,x_\lambda,y_\lambda)\|^2)ds\right)\bigg]d\lambda,
\end{aligned}
$$

where

$$
x_\lambda = \lambda x + (1-\lambda)x_1, \qquad y_\lambda = \lambda y + (1-\lambda)y_1, \quad \lambda \in [0,1].
$$

Set $h = (x - x_1, y - y_1)$, then the Bismut-Elworthy formula yields

$$
\begin{aligned}
H_1(t) = & \int_0^1 \frac{1}{t}\mathbb{E}\bigg[ g\left(Z(t,x_\lambda,y_\lambda)\right) \times \Phi_K\left(\int_0^t (\|u(s,x_\lambda,y_\lambda)\|^2 + \|\theta(s,x_\lambda,y_\lambda)\|^2)ds\right) \\
& \times \int_0^t (Q^{-1/2}DZ(s,x_\lambda,y_\lambda)h, dW(s))\bigg]d\lambda \\
& + 2\int_0^1 \mathbb{E}\bigg[ g\left(Z(t,x_\lambda,y_\lambda)\right) \times \Phi_K'\left(\int_0^t (\|u(s,x_\lambda,y_\lambda)\|^2 + \|\theta(s,x_\lambda,y_\lambda)\|^2)ds\right) \\
& \times \int_0^t (1-\frac{s}{t})(AZ(s,x_\lambda,y_\lambda), DZ(s,x_\lambda,y_\lambda)h)\bigg]d\lambda,
\end{aligned}
$$

where $A : V \times V_1 \to V' \times V_1'$ is the canonical isomorphism of $V \times V_1$ onto $V' \times V_1'$. Let

$$
\tau_\lambda = \inf\left\{ t > 0 : \int_0^t (\|u(s,x_\lambda,y_\lambda)\|^2 + \|\theta(s,x_\lambda,y_\lambda)\|^2)ds \ge 2K \right\}.
$$

Then we have

$$
\begin{aligned}
|H_1(t)| \le C\|g\|_0 \int_0^1 d\lambda \bigg[ & \frac{1}{t}\mathbb{E}\left[\int_0^{t\wedge\tau_\lambda} |Q^{-1/2}DZ(s,x_\lambda,y_\lambda)h|^2 ds\right]^{1/2} \\
& + 2\|\Phi_K'\|_0 \mathbb{E}\left[\left(\int_0^{t\wedge\tau_\lambda} \|\xi^h(s,x_\lambda,y_\lambda)\|^2_{V\times V_1} ds\right)^{1/2} \left(\int_0^t \|Z(s,x_\lambda,y_\lambda)\|^2\right)^{1/2}\right]\bigg],
\end{aligned}
$$

where $\xi^h = DZ \cdot h$. By estimates (4.10) and (4.11), as well as the condition (1.2), we have that

$$\int_0^t |Q^{-1/2} DZ(s, x_\lambda, y_\lambda) h|^2 ds \le C|h|^2.$$

Finally, by estimates (4.2) and (4.10)-(4.13), we obtain

$$\begin{aligned} &|\mathbb{E}\left[g(Z(t,x,y)) - g(Z(t,x_1,y_1))\right]| \\ &\quad \le C\|g\|_0 \delta \left(\frac{\delta}{K} + 2e^{\delta K}(1 + t^{-1/2}) \le \frac{1}{2}\right), \end{aligned} \tag{4.14}$$

for all $x, y, x_1, y_1 \in B_\delta(0)$ when $K$ is appropriately chosen and $\delta$ is small enough. □

With the a priori estimates of Lemmas 4.2-4.4, the next theorem can be obtained by following exactly the same approach (namely, coupling method) as presented in [2].

THEOREM 4.5. *There is a unique invariant measure $\mu$ for semigroup $P_t$.*

**5. Appendix.** DEFINITION 5.1. *Suppose $H$ is a real separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot|$. A linear continuous operator $Q$ is of trace class if it satisfies,*

- *positivity: $(Qx, x) \ge 0$, $x \in H$,*
- *symmetry: $(Qx, y) = (x, Qy)$, $x, y \in H$,*
- *bounded trace: $Tr\ Q := \sum_{k=1}^{\infty} (Qe_k, e_k) < +\infty$ for one (and consequently for all) complete orthonormal system $(e_k)$ in $H$.*

DEFINITION 5.2. *A Markov semigroup $P_t$ on $B_b(H)$ is a mapping*

$$[0, +\infty) \to L(B_b(H)), \quad t \mapsto P_t,$$

*such that*

*(i) $P_0 = 1$, $P_{t+s} = P_t P_s$ for all $t, s \ge 0$.*

*(ii) For any $t \ge 0$ and $x \in H$ there exists a probability measure $\pi_t(x, \cdot) \in \mathcal{P}(H)$ such that*

$$P_t \varphi(x) = \int_H \varphi(y) \pi_t(x, dy) \quad \text{for all } \varphi \in B_b(H).$$

*(iii) For any $\varphi \in C_b(H)$ (resp. $B_b(H)$) and $x \in H$, the mapping $t \mapsto P_t\varphi(x)$ is continuous (resp. Borel).*

DEFINITION 5.3. *Assume $P_t$ represents a Markov semigroup 5.2 on a Hilbert space $H$. A probability measure $\mu \in \mathcal{P}(H)$ is said to be invariant for $P_t$ if*

$$\int_H P_t \varphi d\mu = \int_H \varphi d\mu, \quad \text{for all } \varphi \in B_b(H) \text{ and } t \ge 0,$$

*where $B_b(H)$ is the Banach space of all real-valued Borel bounded mappings defined on $H$ with the norm*

$$\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|.$$

DEFINITION 5.4. *A subset $\Lambda \subset \mathcal{P}(H)$ is said to be tight if there exists an increasing sequence $(K_n)$ of compact sets of $H$ such that*

$$\lim_{n \to \infty} \mu(K_n) = 1 \quad \text{uniformly on } \Lambda,$$

*or, equivalently, if for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon$ such that*

$$\mu(K_\varepsilon) \ge 1 - \varepsilon, \quad \mu \in \Lambda.$$

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