Problem 1. Let \( f : (-\infty, \infty) \to \mathbb{R} \) be continuous and \( \lim_{x \to \infty} f(f(x)) = \infty \). Prove \( \lim_{x \to \infty} |f(x)| = \infty \).

Problem 2. Let \( f_n : X \to \mathbb{R} \), \( n = 1, 2, \ldots \) be a sequence of continuous functions on a metric space \( X \) such that the series \( \sum_{n=1}^{\infty} f_n(x) \) converges for all \( x \in X \) and
\[
\sup_{x \in X} \left( \sum_{n=1}^{\infty} f_n(x)^2 \right)^{1/2} < \infty.
\]
Prove that if a sequence of real numbers \( c_n, n = 1, 2, \ldots \) satisfies \( \sum_{n=1}^{\infty} c_n^2 < \infty \), then the series
\[
\sum_{n=1}^{\infty} c_n f_n(x)
\]
converges everywhere to a continuous function.

Problem 3. Suppose that a set \( A \subset \mathbb{R}^n \) is a union of an increasing family of compact sets \( A = \bigcup_{i=1}^{\infty} A_i \), \( A_1 \subset A_2 \subset \ldots \). Suppose also that there is a compact set \( C \subset \mathbb{R}^n \) such that
\[
\forall i \in \mathbb{N} \forall x \in A \setminus A_i \quad \text{dist}(x, C) < \frac{1}{i}.
\]
Prove that the closure of the set \( A \) satisfies \( \overline{A} \subset A \cup C \).

Remark. Here \( \mathbb{N} \) stands for the set of all positive integers and \( \text{dist}(x, C) = \inf_{y \in C} |x - y| \).

Problem 4. Let \( n \geq 3 \). Consider an \( n \)-times continuously differentiable function \( f \in C^n(\mathbb{R}) \) such that \( f^{(k)}(0) = 0 \), for \( k = 2, 3, \ldots, n-1 \) and \( f^{(n)}(0) \neq 0 \). Clearly, by the mean value theorem for any \( h > 0 \) there is \( 0 < \theta(h) < h \) such that
\[
f(h) - f(0) = hf' \left( \theta(h) \right).
\]
Prove that
\[
\lim_{h \to 0} \frac{\theta(h)}{h} = \left( \frac{1}{n} \right)^{\frac{1}{n-1}}.
\]

Hint: Expand \( f \) and \( f' \) using Taylor's formula.

Problem 5. Prove that \( \lim_{x \to \infty} e^{-x^2} \int_0^x e^{t^2} dt = 0 \).

Problem 6. Let \( f \in C^1(\mathbb{R}) \) be a continuously differentiable function such that \( |f'(x)| \leq 1/2 \) for all \( x \in \mathbb{R} \). Define \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) by
\[
g(x, y) = (x + f(y), y + f(x)).
\]
Prove that
\[
(1) \ g \text{ is a diffeomorphism},
(2) \ g(\mathbb{R}^2) = \mathbb{R}^2,
(3) \ the \ area \ |g([0, 1]^2)| \text{ of the image of the unit square belongs to the interval } [3/4, 5/4].
\]

Hint: Among other tools use the contraction principle.