Preliminary Exam August 2015

Problem 1. Let $f : (-\infty, \infty) \to \mathbb{R}$ be continuous and $\lim_{x\to\infty} f(f(x)) = \infty$. Prove $\lim_{x\to\infty} |f(x)| = \infty$.

Problem 2. Let $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$ be a sequence of continuous functions on a metric space X such that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in X$ and

$$\sup_{x\in X} \left(\sum_{n=1}^{\infty} f_n(x)^2 \right)^{1/2} < \infty.$$

Prove that if a sequence of real numbers c_n, $n = 1, 2, \ldots$ satisfies $\sum_{n=1}^{\infty} c_n^2 < \infty$, then the series

$$\sum_{n=1}^{\infty} c_n f_n(x)$$

converges everywhere to a continuous function.

Problem 3. Suppose that a set $A \subset \mathbb{R}^n$ is a union of an increasing family of compact sets $A = \bigcup_{i=1}^{\infty} A_i$, $A_1 \subset A_2 \subset \ldots$ Suppose also that there is a compact set $C \subset \mathbb{R}^n$ such that

$$\forall i \in \mathbb{N}\, \forall x \in A \setminus A_i \quad \operatorname{dist}(x, C) < \frac{1}{i}.$$

Prove that the closure of the set A satisfies $\overline{A} \subset A \cup C$.

Remark. Here $\mathbb{N}$ stands for the set of all positive integers and $\operatorname{dist}(x, C) = \inf_{y\in C} |x - y|$.

Problem 4. Let $n \geq 3$. Consider an n-times continuously differentiable function $f \in C^n(\mathbb{R})$ such that $f^{(k)}(0) = 0$, for $k = 2, 3, \ldots, n-1$ and $f^{(n)}(0) \neq 0$. Clearly, by the mean value theorem for any $h > 0$ there is $0 < \theta(h) < h$ such that

$$f(h) - f(0) = hf'(\theta(h)).$$

Prove that

$$\lim_{h\to 0} \frac{\theta(h)}{h} = \left(\frac{1}{n}\right)^{\frac{1}{n-1}}.$$

Hint: Expand f and f' using Taylor's formula.

Problem 5. Prove that $\lim_{x\to\infty} e^{-x^2} \int_0^x e^{t^2}\, dt = 0$.

Problem 6. Let $f \in C^1(\mathbb{R})$ be a continuously differentiable function such that $|f'(x)| \leq 1/2$ for all $x \in \mathbb{R}$. Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x, y) = (x + f(y), y + f(x)).$$

Prove that

(1) g is a diffeomorphism,
(2) $g(\mathbb{R}^2) = \mathbb{R}^2$,
(3) the area $|g([0, 1]^2)|$ of the image of the unit square belongs to the interval $[3/4, 5/4]$.

Hint: Among other tools use the contraction principle.