PRELIMINARY EXAM MAY 2016

Problem 1. Prove that if a sequence (a_n) of real numbers converges to a finite limit $\lim_{n\to\infty} a_n = g$, then

$$\lim_{x\to\infty} e^{-x}\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = g.$$

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^4 function such that for all $x, h \in \mathbb{R}$, we have

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x+\frac{1}{3}h)h^2.$$

Show that the fourth derivative $f^{(4)}(x) = 0$ for any $x \in \mathbb{R}$.

Problem 3. Let $f_n : \mathbb{R}^k \to \mathbb{R}^m$ be continuous maps $(n = 1, 2, \dots)$. Let K be a compact subset of $\mathbb{R}^k$. Suppose $f_n \rightrightarrows f$ uniformly on K. Prove that $S = f(K) \cup \bigcup_{n=1}^{\infty} f_n(K)$ is compact.

Problem 4. Let $D = \{(x, y) \mid x^2 + y^2 < 1\}$ be the unit disk in $\mathbb{R}^2$. Let $f, g \in C^2(D)$ be such that g is bounded on D, $f(x, y) \to +\infty$ as $x^2 + y^2 \to 1$, and moreover, $\Delta f = e^f$ and $\Delta g \geq e^g$ at all points of D. Here $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the Laplacian. Show that $f(x, y) \geq g(x, y)$ for any $(x, y) \in D$.

Problem 5. Let $F(x, y) = e^x y^3 + 2x^2y^2 - y\cos x + 2\sin x$, $(x, y) \in \mathbb{R}^2$. Prove that there exist functions $f, g, h \in C^\infty$ defined on an open neighborhood $U \subset \mathbb{R}$ of 0, such that $F(x, f(x)) = F(x, g(x)) = F(x, h(x)) = 0$ and $f(x) < g(x) < h(x)$ for every $x \in U$. Find $f'(0)$, $g'(0)$ and $h'(0)$.

Problem 6. Suppose that smooth functions $f_k : \mathbb{R}^k \to \mathbb{R}$ are defined for $k = 1, 2, \dots 9$. Let $\Phi = (\phi_1, \dots, \phi_{10}) : \mathbb{R}^{10} \to \mathbb{R}^{10}$ be a mapping defined by

$$\begin{aligned}
\phi_1(x_1 \dots, x_{10}) &= x_1\\
\phi_2(x_1 \dots, x_{10}) &= 2x_2 + f_1(x_1)\\
\phi_3(x_1 \dots, x_{10}) &= 3x_3 + f_2(x_1, x_2)\\
&\dots\\
\phi_{10}(x_1 \dots, x_{10}) &= 10\,x_{10} + f_9(x_1 \dots, x_9).
\end{aligned}$$

(1) Prove that Φ is a diffeomorphism of $\mathbb{R}^{10}$ onto an open subset of $\mathbb{R}^{10}$.
(2) Find the volume of $\Phi((-1, 1)^{10})$.