PRELIMINARY EXAMINATION IN ANALYSIS
AUGUST 19, 2016

Problem 1. Given $x_0 > 0$, define a sequence $\{x_n\}$ recursively by $x_n = 3(\sqrt{x_{n-1}+1} - 1)$ for $n \in \mathbb{N}$. For any such x_0, show that $\{x_n\}$ converges, and find its limit.

Problem 2. Prove that there is an increasing sequence of integers $a_1 < a_2 < a_3 < \dots$ such that for every $k \in \mathbb{N}$, the sequence $\{\sin(ka_n)\}_{n=1}^{\infty}$ converges.

Problem 3. For $n \geq 2$ define $f_n \colon [0,1] \to [0,1]$ by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ \frac{n}{n-1}(1-x) & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

Show that $\sum_{n=2}^{\infty} [f_n(x)]^n$ converges pointwise on $[0,1]$ to a function $f(x)$ that is continuous on $(0,1]$, but that the improper integral $\int_0^1 f(x)\, dx$ diverges (the integral is improper at 0).

Problem 4. Let $f : A \to X$ be a mapping between a dense subset $A \subset \mathbb{R}^n$ and a complete metric space (X,d). Assume that $d(f(x), f(y)) \leq |x-y|$ for all $x, y \in A$.

(a) Prove that there is a mapping $F : \mathbb{R}^n \to X$ such that $d(F(x), F(y)) \leq |x-y|$ for all $x, y \in \mathbb{R}^n$ and $F(x) = f(x)$ whenever $x \in A$.

(b) Provide an example showing that the claim in (a) is not true if we do not assume that the space (X,d) is complete.

Problem 5. Suppose $f(x,y)$ is a C^2 function on $\mathbb{R}^2$ such that for some $M > 0$ and all (x,y) in the closed unit disk $\mathbb{D} = \{(x,y) \mid x^2 + y^2 \leq 1\}$,

$$[f_{xx}(x,y)]^2 + 2\,[f_{xy}(x,y)]^2 + [f_{yy}(x,y)]^2 \leq M.$$

If $f(0,0) = f_x(0,0) = f_y(0,0) = 0$, show that

$$\left| \iint_{\mathbb{D}} f(x,y)\, dx\, dy \right| \leq \frac{\pi\sqrt{M}}{4}$$

Problem 6. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ and $\mathbf{v}_i : \mathbb{R} \to \mathbb{R}^n$, $i = 1, 2, \dots, n-1$, be C^∞ smooth functions such that for any $t \in \mathbb{R}$ the vectors

$$\gamma'(t), \mathbf{v}_1(t), \dots, \mathbf{v}_{n-1}(t)$$

form an orthonormal basis of $\mathbb{R}^n$ (here we differentiate γ but **do not** differentiate $\mathbf{v}_i$, $i = 1, 2, \dots, n-1$).

Consider the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi(x_1, \dots, x_n) = \gamma(x_n) + \sum_{i=1}^{n-1} x_i \mathbf{v}_i(x_n).$$

(a) Find the derivative $D\Phi(x_1, \dots, x_n)$;

(b) Prove that Φ is a diffeomorphism in a neighborhood of any point of the form $(0, \dots, 0, x_n)$;

(c) Find the limit

$$\lim_{r \to 0} \frac{|\Phi(B^n(0,r))|}{|B^n(0,r)|},$$

where $B^n(0,r)$ denotes the ball of radius r centered at the origin and $|A|$ stands for the volume of the set A.