The Riemann and Hurwitz zeta functions, Apery's constant and new rational series representations involving $\zeta(2k)$

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A quick overview of the Riemann zeta function.

The *Riemann zeta function* is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \mathrm{Re}\, s > 1.$$

Originally, Riemann zeta function was defined for real arguments. Also, Euler found another formula which relates the Riemann zeta function with prime numbrs, namely

$$\zeta(s) = \prod_{p} \frac{1}{\left(1 - \frac{1}{p^s}\right)},$$

where p runs through all primes $p = 2, 3, 5, \ldots$.

A quick overview of the Riemann zeta function.

Moreover, Riemann proved that the following $\zeta(s)$ satisfies the following integral representation formula:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^{s-1}}{e^u - 1} du, \mathrm{Re}\, s > 1,$$

where $\Gamma(s) = \displaystyle\int_0^\infty t^{s-1} e^{-t} dt$, $\mathrm{Re}\, s > 0$ is the Euler gamma function.

Also, another important fact is that one can extend $\zeta(s)$ from $\mathrm{Re}\, s > 1$ to $\mathrm{Re}\, s > 0$. By an easy computation one has

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^\infty (-1)^{n-1} \frac{1}{n^s},$$

and therefore we have

A quick overview of the Riemann function.

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}, \operatorname{Re} s > 0, s \neq 1.$$

It is well-known that ζ is analytic and it has an analytic continuation at $s = 1$. At $s = 1$ it has a simple pole with residue 1. We have

$$\lim_{s \to 1} (s-1)\,\zeta(s) = 1.$$

Let us remark that the alternating zeta function is called the *Dirichlet eta function* and it is defined as

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}, \operatorname{Re} s > 0, s \neq 1.$$

A quick overview of the Hurwitz zeta function.

Another important function is the *Hurwitz (generalized) zeta function* defined by

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},\, \mathrm{Re}\, s>1, a\neq 0,-1,-2,\ldots .$$

As the Riemann zeta function, Hurwitz zeta function is analytic over the whole complex plane except $s=1$ where it has a simple pole. Also, from the two definitions, one has

$$\zeta(s)=\zeta(s,1)=\frac{1}{2^s-1}\zeta\left(s,\frac{1}{2}\right)=1+\zeta(s,2).$$

A quick overview of the Hurwitz zeta function.

It can also be extended by analytic continuation to a meromorphic function defined for all complex numbers $s \neq 1$. At $s = 1$ it has a simple pole with residue 1. The constant term is given by

$$\lim_{s \to 1} \left(\zeta(s, a) - \frac{1}{s-1} \right) = -\frac{\Gamma'(a)}{\Gamma(a)} = -\Psi(a),$$

where Ψ is the digamma function. Also, the Hurwitz zeta function is related to the *polygamma function*,

$$\Psi_m(z) = (-1)^{m+1} m! \zeta(m+1, z).$$

A quick overview of the Dirichlet beta function.

Last but not least, we define the *Dirichlet beta function* as

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \operatorname{Re} s > 0.$$

Alternatively, one can express the beta function by the following formula:

$$\beta(s) = \frac{1}{4^s}\left(\zeta(s, 1/4) - \zeta(s, 3/4)\right).$$

Equivalently, $\beta(s)$ has the following integral representation:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-t}}{1+e^{-2t}} dt.$$

A quick overview of the Dirichlet beta function.

Note that $\beta(2) = G$ (Catalan's constant), $\beta(3) = \frac{\pi^3}{32}$, and

$$\beta(2n+1) = \frac{(-1)^n E_{2n} \pi^{2n+1}}{4^{n+1}(2n)!}$$

where E_n are the Euler numbers in the Taylor series

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n.$$

Other special values include

$$\beta(0) = \frac{1}{2}, \beta(1) = \frac{\pi}{4}, \beta(-k) = \frac{E_k}{2}.$$

What is known about the values of $\zeta(s)$ at integers?

- $\zeta(-2n) = 0$ for $n = 1, 2, \dots$ (trivial zeros)
- $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$; with $\zeta(-1) = -\frac{1}{12}$
- The values $\zeta(2n)$, for $n = 1, 2, \dots$ have been found by Euler in 1740
- The values $\zeta(-2n+1)$, for $n = 1, 2, \dots$ can be evaluated in terms of $\zeta(2n)$. In fact, we have

$$\zeta(-2n+1) = 2(2\pi)^{2n}(-1)^n(2n-1)!\zeta(2n).$$

- There is a mystery about $\zeta(2n+1)$ values
- $\zeta(0) = -\frac{1}{2}$
- $\zeta(1)$ does not exist, but one has the following

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma.$$

Another quick look at $\zeta(2n)$ and $\zeta(2n+1)$

In 1734, Euler produced a sensation when he discovered that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Later, the same Euler generalized the above formula,

$$\zeta(2n) = (-1)^{n+1} \cdot \frac{B_{2n} 2^{2n-1} \pi^{2n}}{(2n)!},$$

where the coefficients B_n are the so-called Bernoulli numbers and they satisfy

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, |z| < 2\pi.$$

Another quick look at $\zeta(2n)$ and $\zeta(2n+1)$

An elementary but sleek proof of Euler's result was recently given in

E. De Amo, M. Diaz Carrillo, J. Hernandez-Sanchez, Another proof of Euler's formula for $\zeta(2k)$, *Proc. Amer. Math. Soc.* **139** (2011), 1441–1444.

The authors proved Euler's formula using the Taylor series for the tangent function and Fubini's theorem.
Unlike $\zeta(2n)$, the values $\zeta(2n+1)$ are still mysterious! One of the most important results was produced by Roger Apery in 1979, when he proved that $\zeta(3)$ is irrational by using the "fast converging" series representation

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3\binom{2n}{n}}.$$

Another quick look at $\zeta(2n)$ and $\zeta(2n+1)$

Amazingly, there exist similar formulas for $\zeta(2)$ and $\zeta(4)$, namely

$$\zeta(2) = 3\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}}, \zeta(4) = \frac{36}{17}\sum_{n=1}^{\infty} \frac{1}{n^4\binom{2n}{n}}.$$

Recently, other substantial results were obtained. In 2002, K. Ball and T. Rivoal proved the following

K. Ball, T. Rivoal, Irrationalite d'une infinite de la fonction zetaaux entiers impairs, *Invent. Math.* **146** (2001), 193–207.

Theorem

There are infinitely many irrational values of the Riemann zeta function at odd positive integers. Moreover, if
$N(n) = \#\{\text{irrational numbers among } \zeta(3), \zeta(5), \ldots, \zeta(2n+1)\}$,
then $N(n) \geq \dfrac{1}{2(1+\log 2)} \log n$ *for large* n.

Another quick look at $\zeta(2n)$ and $\zeta(2n+1)$

Other remarkable results in this direction are given by Rivoal (2001) and Zudilin (2001),

Theorem

At least four numbers $\zeta(5), \zeta(7), \ldots, \zeta(21)$ *are irrational.*

and

W. Zudilin, One of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational, *Russ. Math. Surv.* **56** (2001), 193–206.

Theorem

At least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational.*

Some Taylor series representations

For the sake of completeness we display the Taylor series for the tangent, cotangent, secant and cosecant functions:

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n}}{(2n)!} x^{2n-1}, \; |x| < \frac{\pi}{2} \tag{1}$$

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n} B_{2n}}{(2n)!} x^{2n-1}, \; |x| < \pi \tag{2}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^{n} E_{2n}}{(2n)!} x^{2n}, \; |x| < \frac{\pi}{2} \tag{3}$$

$$\csc x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2 (2^{2n-1} - 1) B_{2n}}{(2n)!} x^{2n-1}, \; |x| < \pi \tag{4}$$

Clausen integral

The *Clausen function (Clausen integral)*, introduced by Thomas Clausen in 1832, is a transcendental special function of single variable and it is defined by

$$\mathrm{Cl}_2(\theta) := \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^2} = -\int_0^{\theta} \log\left(2\sin\left(\frac{x}{2}\right)\right)\, dx.$$

It is intimately connected with the polylogarithm, inverse tangent integral, polygamma function, Riemann zeta function, Dirichlet eta function, and Dirichlet beta function.
Some well-known properties of the Clausen function include periodicity in the following sense:

$$\mathrm{Cl}_2(2k\pi \pm \theta) = \mathrm{Cl}_2(\pm\theta) = \pm\,\mathrm{Cl}_2(\theta).$$

Clausen integral

Moreover, it is quite clear from the definition that $\mathrm{Cl}_2(k\pi) = 0$ for k integer. For example, for $k = 1$ we deduce

$$\int_0^{\pi} \log\left(2\sin\left(\frac{x}{2}\right)\right)\, dx = 0, \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2.$$

By periodicity we have $\mathrm{Cl}_2\left(\frac{\pi}{2}\right) = -\mathrm{Cl}_2\left(\frac{3\pi}{2}\right) = G$, where G is the Catalan constant defined by

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159...$$

Clausen integral. Evaluation of some elementary integrals

More generally, one can express the above integral as the following:

$$\int_0^\theta \log(\sin x)\, dx = -\frac{1}{2}\,\mathrm{Cl}_2(2\theta) - \theta \log 2,$$

$$\int_0^\theta \log(\cos x)\, dx = -\frac{1}{2}\,\mathrm{Cl}_2(\pi - 2\theta) - \theta \log 2,$$

$$\int_0^\theta \log(1 + \cos x)\, dx = 2\,\mathrm{Cl}_2(\pi - \theta) - \theta \log 2,$$

and

$$\int_0^\theta \log(1 + \sin x)\, dx = 2G - 2\,\mathrm{Cl}_2\left(\frac{\pi}{2} + \theta\right) - \theta \log 2.$$

Clausen acceleration formula

Theorem

We have the following representation for the Clausen function $\mathrm{Cl}_2(\theta)$,

$$\frac{\mathrm{Cl}_2(\theta)}{\theta} = 1 - \log|\theta| + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2\pi)^{2n} n(2n+1)} \theta^{2n}, |\theta| < 2\pi. \qquad (5)$$

Sketch of the proof. Integrating by parts the function $xy\cot(xy)$ and using the product formulas for the sine function, we have

$$\int_0^{\frac{\pi}{2}} xy\cot(xy)dx = \frac{\pi}{2}\log\left(\frac{\pi y}{2}\right) - \frac{\pi}{2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\frac{y^2}{4k^2}\right)^n}{n} + \frac{\pi}{2}\log 2 +$$

Clausen acceleration formula

$$+\frac{1}{2y}\mathrm{Cl}_2(\pi y).$$

On the other hand, by the Taylor series for the cotangent function, we obtain

$$\int_0^{\frac{\pi}{2}} xy\cot(xy)\,dx = \frac{\pi}{2} - 2\sum_{n=1}^{\infty}\frac{\zeta(2n)\left(\frac{\pi}{2}\right)^{2n+1}}{\pi^{2n}(2n+1)}y^{2n}.$$

Therefore, we obtain

$$\frac{\pi}{2}-2\sum_{n=1}^{\infty}\frac{\zeta(2n)\left(\frac{\pi}{2}\right)^{2n+1}}{\pi^{2n}(2n+1)}y^{2n} = \frac{\pi}{2}\log(\pi y)-\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}y^{2n}+\frac{1}{2y}\mathrm{Cl}_2(\pi y).$$

Equate the coefficients of y^{2n} and we obtain our formula.$\square$

Clausen acceleration formula. Some remarks.

In particular case of $\theta = \frac{\pi}{2}$, using the fact that $\mathrm{Cl}_2(\frac{\pi}{2}) = G$, we obtain

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)16^n} = \frac{2G}{\pi} - 1 + \log\left(\frac{\pi}{2}\right). \tag{6}$$

Recently, Wu, Zhang and Liu developed the following representation for the Clausen function,

J. Wu, X. Zhang, D. Liu, An efficient calculation of the Clausen functions, *BIT Numer. Math.* **50** (2010), 193–206.

$$\mathrm{Cl}_2(\theta) = \theta - \theta \log\left(2 \sin \frac{\theta}{2}\right) - \sum_{n=1}^{\infty} \frac{2\zeta(2n)}{(2n+1)(2\pi)^{2n}} \theta^{2n+1}, |\theta| < 2\pi. \tag{7}$$

Clausen acceleration formula. Some well-known representation for $\zeta(3)$.

It is interesting to see that integrating the above formula from 0 to $\pi/2$ we have the following representation for $\zeta(3)$ due to Choi, Srivastava and Adamchik,

H. M. Srivastava, M. L. Glasser, V. S. Adamchik, Some definite integrals associated with the Riemann zeta function, *J. Z. Anal. Anwendungen.* **19** (2000), 831–846.

$$\zeta(3) = \frac{4\pi^2}{35}\left(\frac{1}{2} + \frac{2G}{\pi} - \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(n+1)(2n+1)16^n}\right). \qquad (8)$$

Also, Strivastava, Glasser and Adamchik derive series representations for $\zeta(2n+1)$ by evaluating the integral $\int_0^{\pi/\omega} t^{s-1} \cot t dt$, $s, \omega \geq 2$ integers in two different ways. One of the ways involves the generalized Clausen functions. When they are evaulated in terms of $\zeta(2n+1)$ one obtains the following formula for $\zeta(3)$,

$$\zeta(3) = \frac{2\pi^2}{9}\left(\log 2 + 2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+3)4^n}\right). \tag{9}$$

Other representations for $\zeta(3)$

Other representations for Apery's constant are given by Cvijovic and Klinowski,

D. Cvijovic, J. Klinowski, New rapidly convergent series representations for $\zeta(2n+1)$, *Proc. Amer. Math. Soc.* **125** (1997), 1263–1271.

$$\zeta(3) = -\frac{\pi^2}{3}\sum_{n=0}^{\infty}\frac{(2n+5)\zeta(2n)}{(2n+1)(2n+2)(2n+3)2^{2n}}. \tag{10}$$

and

$$\zeta(3) = -\frac{4\pi^2}{7}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}. \tag{11}$$

New series representations for Apery's constant $\zeta(3)$

Theorem

$$\zeta(3) = \frac{4\pi^2}{35}\left(\frac{3}{2} - \log\left(\frac{\pi}{2}\right) + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)(2n+1)16^n}\right), \quad (12)$$

$$\zeta(3) = -\frac{64}{3\pi}\beta(4) + \frac{8\pi^2}{9}\left(\frac{4}{3} - \log\left(\frac{\pi}{2}\right) + 3\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+3)16^n}\right), \quad (13)$$

and

$$\zeta(3) = -\frac{64}{3\pi}\beta(4) + \frac{16\pi^2}{27}\left(\frac{1}{2} + \frac{3G}{\pi} - 3\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+3)16^n}\right), \quad (14)$$

Ideas of the proof

The main ingredients in the proof of the above theorem are the following:

- Fubini's theorem, Clausen acceleration formulas and $\int_0^{\pi/4} u \log(\sin u) du = \frac{35}{128}\zeta(3) - \frac{\pi G}{8} - \frac{\pi^2}{32} \log 2$ to find
$$\int_0^{\pi/2} \mathrm{Cl}_2(y) = \frac{35}{32}\zeta(3).$$
- It can be proven using Fubini's theorem a formula for $\int_0^{\pi^2/4} \mathrm{Cl}_2(\sqrt{y})dy$ which combined with the polygamma formula related to the Hurwitz zeta function give us
$$\int_0^{\pi^2/4} \mathrm{Cl}_2(\sqrt{y})dy = \frac{3\pi}{32}\zeta(3) + 2\beta(4).$$

Rational series representation involving $\zeta(2n)$

We shall call *rational ζ-series* of a real number x, the following representation:

$$x = \sum_{n=2}^{\infty} q_n \zeta(n, m),$$

where q_n is a rational number and $\zeta(n, m)$ is the Hurwitz zeta function. For $m > 1$ integer, one has

$$x = \sum_{n=2}^{\infty} q_n \left(\zeta(n) - \sum_{j=1}^{m-1} j^{-n} \right).$$

J. M. Borwein, D. M. Bradley, R. E. Crandall, Computational strategies for the Riemann zeta function, *J. Comp. Appl. Math.* **121** (2000), 247–296.

Rational series representation involving $\zeta(2n)$. Examples

In the particular case $m = 2$, one has the following series representations:

$$1 = \sum_{n=2}^{\infty} (\zeta(n) - 1)$$

$$1 - \gamma = \sum_{n=2}^{\infty} \frac{1}{n} (\zeta(n) - 1)$$

$$\log 2 = \sum_{n=2}^{\infty} \frac{1}{n} (\zeta(2n) - 1),$$

where γ is the Euler-Mascheroni constant.

New rational series representation involving $\zeta(2n)$

Theorem

The following representation is true

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}\binom{2n}{m}=\begin{cases}\dfrac{1}{m} & m\,\text{odd},\\[2ex] \dfrac{1}{m}\left(2\zeta(m)\left(1-\dfrac{1}{2^m}\right)-1\right) & m\,\text{even}.\end{cases} \tag{15}$$

Some corollaries

In particular cases we obtain the following

Corollary

We have

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)4^n} = \log \pi - 1. \tag{16}$$

and

Some corollaries

Corollary

We have the following series representations

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{4^n} = \frac{1}{2}, \tag{17}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)(2n-1)(2n-2)}{4^n} = 1, \tag{18}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)(2n-1)}{4^n} = \frac{\pi^2}{8} - \frac{1}{2}, \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)n}{4^n} = \frac{\pi^2}{16}, \tag{20}$$

New rational series representation involving $\zeta(2n)$

Theorem

We have the following series representation

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n16^n} \binom{2n}{m} = \begin{cases} \dfrac{1}{m}\left(1-\beta(m)\right) & m \text{ odd}, \\[2ex] \dfrac{1}{m}\left(\zeta(m)\left(1-\dfrac{1}{2^m}\right)-1\right) & m \text{ even}, \end{cases} \tag{22}$$

Some corollaries again

From the previous theorem we recover some well-known rational series representations for π

Corollary

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n16^n} = \log\left(\frac{\pi}{2\sqrt{2}}\right), \tag{23}$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{16^n} = \frac{4-\pi}{8}. \tag{24}$$

Some corollaries again and again

Corollary

We have the following series

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left(1 - \frac{1}{4^n}\right) \binom{2n}{2k} = \frac{\zeta(2k)}{2k} \left(1 - \frac{1}{4^k}\right), \tag{25}$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left(1 - \frac{1}{4^n}\right) \binom{2n}{2k+1} = \frac{\beta(2k+1)}{2k+1}. \tag{26}$$

And one more...

Corollary

We have

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)(2n-1)}{16^n} = \frac{\pi^2}{16} - \frac{1}{2}, \tag{27}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)(2n-1)(2n-2)}{16^n} = 1 - \frac{\pi^3}{96}, \tag{28}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)n}{16^n} = \frac{\pi}{16}\left(\frac{\pi}{2} - 1\right), \tag{29}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)n^2}{16^n} = \frac{\pi}{32}\left(\frac{3\pi}{2} - \frac{\pi^2}{4} - 1\right). \tag{30}$$

Thank you for your attention!!!