“THE MAZUR PRODUCT OF CONVERGENT SEQUENCES IS NOT ONTO. 3.”

FUNCTIONAL ANALYSIS SEMINAR

TUESDAY 25/FEBRUARY/2020, 3:00 PM - 3:50 PM
ROOM 321 THACKERAY

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Abstract. We present a discussion of a 2009 solution of Mazur’s Problem 8 from the Scottish Book by [L/Dan Radelet, JMAA 350, pp. 384-392]. It is well known that the Cesàro mapping $C$ on the space of convergent sequences of complex numbers $c$, 

$$C : x = (x_n)_{n \geq 0} \mapsto \left( \frac{x_0 + x_1 + \cdots + x_n}{n + 1} \right)_{n \geq 0},$$

preserves the limit $L = \lambda(x) := \lim_{n \to \infty} x_n$, but is not onto.

Put differently, $C$ maps $c$ into $c$, $\lambda(Cx) = \lambda(x)$, and there exists $z = (z_n)_{n \geq 0}$ in $c$ such that $z \neq Cx$, for all $x \in c$. Indeed, 

$$z = (1, 0, 1/3, 0, 1/5, 0, 1/7, 0, \ldots)$$

is such a sequence.

In the Scottish Book, Stanisław Mazur considered an extension of the Cesàro mapping to a bilinear product $\boxtimes$ on $c$: for all $x, y \in c$, 

$$x \boxtimes y := \left( \frac{x_0 y_n + x_1 y_{n-1} + \cdots + x_n y_0}{n + 1} \right)_{n \geq 0}.$$ 

It turns out that for all $x, y \in c$, $x \boxtimes y \in c$ and 

$$\lambda(x \boxtimes y) = \lambda(x) \lambda(y).$$
Note also that $x \boxplus I = Cx$, where $I := (1, 1, 1, \ldots, 1, \ldots)$. In the Scottish Book, Problem 8, Professor Mazur asked whether the mapping $\boxplus : c \times c \rightarrow c$ is onto? The prize for solution, written in the margin, was 5 small beers.

This question was solved in the negative in the 1980’s: 

There exists $z \in c$ such that $z \neq x \boxplus y$, for all $x, y \in c$. It was solved independently by P.P.B. Eggermont and Y.J. Leung, as well as S. Kwapień and A. Pełczyński, and also by V.V. Peller.

Later, L/Dan Radelet (2009) came up with another solution to Mazur’s problem 8, by solving a more general question. Note that to solve Mazur’s problem in the negative, it is sufficient to showing that $\boxplus : c \times c_0 \rightarrow c_0$ is not onto. Here, $c_0$ is the space of all real sequences that converge to zero. L/Radelet showed that 

There exists $z \in c_0$ such that $z \neq x \boxplus y$, for all $x \in \mathcal{H}^2$ and for all $y \in \mathcal{H}^2_0$, where $\mathcal{H}^2$ is a larger space than $c$ and $\mathcal{H}^2_0$ is a larger space than $c_0$.

We will discuss all this, and the ideas behind our solution (including plenty of illustrative examples), over the span of a few Functional Analysis Seminars.