Summary for Dec 15

The only closed constant curvature space curves which I knew in 2004 were made from pieces of circles and helices. The Frenet equations allow to construct space curves of constant curvature κ by prescribing a torsion function $\tau(s)$. For closed curves one needs periodic torsion functions, for example Fourier polynomials. If one chooses these functions so that they are skew symmetric with respect to their zeros, $\tau(a-s) = -\tau(a+s)$, then the resulting curves have the normal planes at these points as planes of mirror symmetry. If adjacent symmetry planes have angles such as $\pi/3$, then the curves are forced by their symmetries to be closed. This gives the first collection of new examples.

If the torsion functions are even with respect to their extremal points, i.e. $\tau(a-s) = \tau(a+s)$, then the resulting curves have their principal normals at these points as symmetry axes for 180° rotations. If two such adjacent symmetry normals are coplanar and intersect under rational angles $(\pi p/q)$, then the curves are again forced by their symmetries to close up. Therefore one can hope to get examples by solving a 2-parameter problem.

This is made simple by an *observation* which I cannot prove: The distance of adjacent symmetry normals depends in a surprisingly monotone way on the constant term in the Fourier polynomial $\tau(s) = b_0 + b_1 \sin(s) + b_3 \sin(3s)$. This allows to consider $b_0 = b_0(\kappa, b_1, b_3)$, such that the symmetry normals of the resulting curves are coplanar and hence intersect all in one point. Therefore we have again to solve a 1-parameter problem by choosing κ, b_1, b_3 in such a way that adjacent symmetry normals intersect with an rational angle. This gives a wealth of new examples.

The evolutes of such curves have also constant curvature κ, but they have singularities at the zeros of τ. This led to a search for closed examples with $\tau(s) > 0$. A (2-11)-torus knot showed up and suggested to look for examples on tori. Easy ones are again found by symmetries and more complicated ones as solutions of intermediate value problems. The formulas work also on cylinders and revealed easier examples than all the previous ones!

Then E. Tjaden suggested to look for examples which are congruent to their evolutes. They were found by modifying the Frenet equations. The $(2, 2n+1)$-torus knots among them are in fact their own evolutes.

This story will be told along with pictures illustrating all steps, including the *observation* above which we used but could not prove. Since the shape of space curves is difficult to be correctly deduced from planar images, most images will be red-green anaglyphs. They can be looked at without red-green glasses, but without giving the 3D impression.