Geodesics in the Heisenberg Group $\mathbb{H}^n$

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Lie Group structure

The Heisenberg Group $\mathbb{H}^1$ is $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ with an additional Lie Group structure.

The Lie Group multiplication is defined for any $(z, t), (z', t') \in \mathbb{H}^1$ as

$$(z, t) * (z', t') = (z + z', t + t' + 2\mathrm{Im}(z\bar{z}'))$$

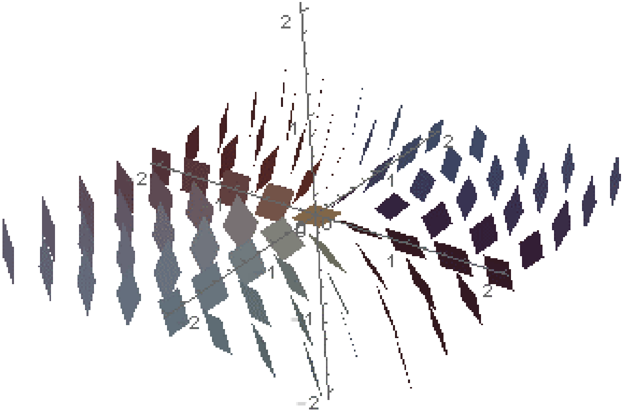
and the Lie algebra $\mathfrak{g}$ of $\mathbb{H}^1$ has the basis of left invariant vector fields at any $p = (x, y, t)$

$$X(p) = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t} \quad Y(p) = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t} \quad T(p) = \frac{\partial}{\partial t}.$$

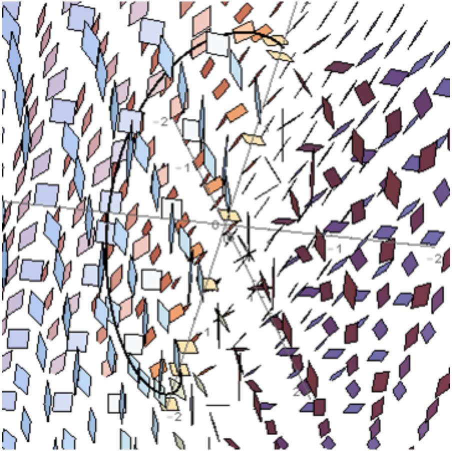
We call $H\mathbb{H}^1 = \mathrm{span}\{X, Y\}$ the *horizontal distribution* on $\mathbb{H}^1$ and call $H_p\mathbb{H}^1$ the horizontal space at p.

We say that an absolutely continuous curve $\Gamma : [0, S] \to \mathbb{H}^1$ is *horizontal* if $\dot{\Gamma}(s) \in H_{\Gamma(s)}\mathbb{H}^1$ for almost every $s \in [0, S]$.

The horizontal distribution



The horizontal distribution



(All pictures and video created by Jake Mirra.)

Horizontal curves

In other words, an a.c. curve $\Gamma = (x, y, t) : [0, S] \to \mathbb{H}^1$ is horizontal if and only if

$$\dot{t}(s) = 2(\dot{x}(s)y(s) - \dot{y}(s)x(s))$$

for almost every $s \in [0, S]$.

From this we can conclude that

$$t(S) - t(0) = 2\int_0^S (\dot{x}(s)y(s) - \dot{y}(s)x(s))\, ds.$$

That is, if we are given a curve $\gamma = (x, y) : [0, S] \to \mathbb{R}^2$, the horizontal curve $\Gamma = (x, y, t)$ is uniquely determined (up to its starting height $t(0)$). We call Γ the *horizontal lift* of γ.

Length and distance in $\mathbb{H}^1$

The length in $\mathbb{H}^1$ of a horizontal curve Γ equals

$$\ell_H(\Gamma) = \int_0^S \sqrt{\dot{x}(s)^2 + \dot{y}(s)^2}\, ds = \ell_E(\gamma).$$

where γ is its projection onto the first two coordinates.

Define the usual Carnot-Caratheodory metric d_{cc} on $\mathbb{H}^1$ so that $d_{cc}(p,q)$ equals

$$\inf\{\ell_H(\Gamma) \,|\, \Gamma : [0,S] \to \mathbb{H}^1 \text{ is horizontal and } \Gamma(0) = p, \Gamma(S) = q\}.$$

Geodesics in $\mathbb{H}^1$

A *geodesic* between points $p, q \in \mathbb{H}^1$ is a horizontal curve of shortest length connecting p and q. What do geodesics look like in $\mathbb{H}^1$?

Suppose $\Gamma = (\gamma, t)$ is a horizontal curve connecting the origin to a point $(0, 0, T)$ on the t-axis. (For simplicity, assume its projection γ has no self intersections.) Recall the equation for the change T in the height of the curve:

$$T = t(S) - t(0) = 2\int_0^S (\dot{x}(s)y(s) - \dot{y}(s)x(s))\,ds.$$

By Green's theorem, $T = -4A$ where A is the area enclosed by the closed curve γ.

Geodesics in $\mathbb{H}^1$

Recall also that $\ell_H(\Gamma) = \ell_E(\gamma)$. Hence, finding a geodesic connecting the origin to the point $(0, 0, T)$ is equivalent to finding the shortest curve in $\mathbb{R}^2$ which encloses a fixed area. (Since the horizontal lift of a curve in $\mathbb{R}^2$ is unique.)

The isoperimetric inequality states that this curve must be a circle.

Theorem

A horizontal curve connecting the origin to the point $(0, 0, T)$ in $\mathbb{H}^1$ is a geodesic if an only if its projection onto its first two coordinates is a circle.

The Heisenberg Group $\mathbb{H}^n$ is $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ with an additional Lie Group structure.

The Lie Group multiplication is defined for any $(z,t),(z',t') \in \mathbb{H}^n$ as

$$(z,t) * (z',t') = (z+z', t+t'+2\mathrm{Im}(z \cdot \bar{z}'))$$

and the Lie algebra $\mathfrak{g}$ of $\mathbb{H}^n$ has the basis of left invariant vector fields at any $p = (x_1,\ldots,x_n,y_1,\ldots,y_n,t)$

$$X_j(p) = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t} \quad Y_j(p) = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t} \quad T(p) = \frac{\partial}{\partial t}.$$

for any $j = 1,\ldots,n$.

We call $H\mathbb{H}^n = \mathrm{span}\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$ the *horizontal distribution* on $\mathbb{H}^n$ and call $H_p\mathbb{H}^n$ the horizontal space at p.

We say that an absolutely continuous curve $\Gamma : [0,S] \to \mathbb{H}^n$ is *horizontal* if $\dot{\Gamma}(s) \in H_{\Gamma(s)}\mathbb{H}^n$ for almost every $s \in [0,S]$.

Horizontal Curves in $\mathbb{H}^n$

In other words, an a.c. curve $\Gamma = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) : [0, S] \to \mathbb{H}^n$ is horizontal if and only if

$$\dot{t}(s) = 2\sum_{j=1}^{n}(\dot{x}_j(s)y_j(s) - \dot{y}_j(s)x_j(s))$$

for almost every $s \in [0, S]$.

From this we can conclude that

$$t(S) - t(0) = 2\sum_{j=1}^{n}\int_0^S (\dot{x}_j(s)y_j(s) - \dot{y}_j(s)x_j(s))\, ds.$$

That is, if we are given a curve $\gamma = (x_1, \ldots, x_n, y_1, \ldots, y_n) : [0, S] \to \mathbb{R}^{2n}$, the horizontal curve $\Gamma = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ is uniquely determined (up to its starting height $t(0)$). We call Γ the *horizontal lift* of γ.

Length and distance in $\mathbb{H}^n$

Similar to before, the length in $\mathbb{H}^n$ of a horizontal curve Γ equals

$$\ell_H(\Gamma) = \int_0^S \sqrt{\sum_{j=1}^n (\dot{x}_j(s)^2 + \dot{y}_j(s)^2)}\, ds = \ell_E(\gamma).$$

where γ is its projection onto the first $2n$ coordinates.

Again define d_{cc} on $\mathbb{H}^n$ so that $d_{cc}(p,q)$ equals

$$\inf\{\ell_H(\Gamma) \,|\, \Gamma : [0,S] \to \mathbb{H}^n \text{ is horizontal and } \Gamma(0) = p, \Gamma(S) = q\}.$$

Geodesics in $\mathbb{H}^n$

Unfortunately, the argument that we used in $\mathbb{H}^1$ to prove that geodesics must be the horizontal lifts of circles does not apply directly in $\mathbb{H}^n$. The isoperimetric inequality applies to curves in $\mathbb{R}^2$ but not for curves in $\mathbb{R}^{2n}$ when $n > 1$.

However, there is a proof of the isoperimetric inequality by Adolf Hurwitz which uses Fourier series to compare the length of a curve and the area it encloses.

Geodesics in $\mathbb{H}^n$

The integral equation for the change in the height of a horizontal curve involves a pair of L^2 inner products, and the equation for the length of a curve involves a pair of L^2 norms. Thus we can apply Parseval's identity to compare these two equations to one another and find an inequality relating the length of a horizontal curve to the area it encloses. This inequality allows us to determine the shape of geodesics in $\mathbb{H}^n$ as before.

Theorem

A horizontal curve connecting the origin to a point $(0, \ldots, 0, T)$ in $\mathbb{H}^n$ is a geodesic if and only if the projection of the curve to each $x_j y_j$-plane is a circle with constant speed.

References

Piotr Hajłasz, Scott Zimmerman (2015)
Geodesics in the Heisenberg Group $\mathbb{H}^n$ by way of Fourier series.
Submitted for publication.

The End