Analysis of time filters used with the leapfrog scheme

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Abstract

We present the linear analysis of recent time filters used in numerical weather prediction. We focus on the accuracy and the stability of the leapfrog scheme combined with the Robert–Asselin–Williams filter, the higher-order Robert–Asselin type time filter, the composite-tendency Robert–Asselin–Williams filter and a more discriminating filter.

Keywords: Numerical weather prediction, Leapfrog method, Time filters

1 Introduction

The *leapfrog* (LF) time-stepping scheme emerged, from the early years of numerical weather prediction, as the method of choice and is still popular for a number of reasons. Perhaps the most important attribute of the leapfrog scheme is that it preserves exactly the amplitude of a pure oscillation. The dissipative characteristics of other time integration schemes are generally too strong, while the absence of computational damping of leapfrog scheme is especially desirable for long-time integrations. Another feature of the leapfrog

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method is efficiency, namely, it evaluates the right-hand side of the meteorological tendency equations only once per time step, in contrast with most other schemes. The leapfrog scheme applied to a generic differential equation

$$\frac{du}{dt} = F(u)$$

is given by

$$u^{n+1} = u^{n-1} + 2\Delta t F(u^n),$$

where Δt is the time step and u^n is the approximated solution at time $t^n = n\Delta t$.

The leapfrog method is a three-time-level scheme, and when applied to a simple set of linear differential equations, it generates two modes of motion. One is the *physical* mode, which contributes to the true solution, while the other one is the *computational* mode, which is merely artificial and has no relation to the differential equations that are being integrated. The computational mode of the leapfrog scheme is undamped in linear problems, meaning that it preserves the amplitude in each time step. In nonlinear problems, however, the nonlinear terms introduce couplings between the physical and computational modes which may amplify the computational mode. In short-time simulations of weather and climate, the growth of the computational mode is generally hard to detect, but when long-time integrations are considered, the computational mode dominates the solution.

One possible approach to control the leapfrog scheme's computational mode is to periodically use a two-time-level scheme, e.g., a Matsuno step after every 11 leapfrog steps [21]. The idea is to reset the amplitude to zero periodically, so it never becomes large enough/problematic. Another technique is to use different explicit time-stepping schemes, e.g., the second-order Adams-Bashforth method [18], the third-order Adam-Bashforth [8], the leapfrog-trapezoidal method [14, 33] or the Magazenkov method [19].

The ubiquitous strategy in atmospheric models, for controlling the leapfrog scheme's computational mode, is the non-intrusive implementation of a time filter after each leapfrog time step. Robert [25] designed such a filter, which Asselin [3] analyzed and proved to effectively damp the computational mode of the leapfrog scheme. This time filter is referred to as the Robert–Asselin (RA) filter. The RA-filtered leapfrog scheme is defined by

$$v^{n+1} = u^{n-1} + 2\Delta t F(v^n),$$

$$u^n = v^n + \frac{\nu}{2}\left(v^{n+1} - 2v^n + u^{n-1}\right),$$

where v and u denote the unfiltered (provisional) and filtered (definitive) variables, respectively. The dimensionless parameter $\nu \in [0, 1]$ determines the strength of the filter.

The accuracy and stability properties of the RA filter were investigated in [4, 12, 6, 10, 26, 7, 24, 5, 28, 13]. Currently, the RA filter is used in operational numerical weather prediction models, atmospheric general circulation models for climate simulation, ocean general circulation models, and models of fluids in rotating annulus laboratory experiments, etc. A comprehensive list of atmospheric models with RA filter can be found in [28]. Unfortunately, the RA filter also damps the physical mode. As a result, the formal second-order accuracy of the leapfrog scheme is reduced to first order, and can degrade the accuracy of model simulations. Therefore, physical quantities (e.g., energy) conserved by the time-continuous equations are not necessarily conserved by time-discretized equations when the filter is used.

Because the RA-filtered leapfrog scheme is widely used in legacy codes for atmospheric models, non-intrusive and simple-to-implement improvements of RA appear attractive, in order to avoid the significant programming undertaking. Williams [28] proposed a modification of the RA filter, which combined with the leapfrog scheme is

$$\begin{aligned} w^{n+1} &= u^{n-1} + 2\Delta t F(v^n), \\ u^n &= v^n + \frac{\nu\alpha}{2}(w^{n+1} - 2v^n + u^{n-1}), \\ v^{n+1} &= w^{n+1} - \frac{\nu(1-\alpha)}{2}(w^{n+1} - 2v^n + u^{n-1}), \end{aligned}$$

where w, v, and u denote the unfiltered, once filtered, and twice filtered variables, respectively. The parameter ν is as in the RA filter, and the new dimensionless parameter $\alpha \in [0.5, 1]$. Linear analysis shows that ν plays a role in controlling the computational mode of the leapfrog scheme, while α is the remedy in restoring accuracy. The filter is now referred to as the Robert–Asselin–Williams (RAW) filter. It reduces the negative impact of the RA filter on the physical mode and increases the numerical accuracy to second order, at the price of a slight instability. The filter has been implemented and studied in [27, 1, 23, 32, 20, 31, 13], and its behavior in implicit-explicit (IMEX) integrations was analyzed in [29].

Later, Williams [30] proposed two methods for further improving the RAW-filtered leapfrog scheme. The first algorithm is a combination of the RAW filter with a composite-tendency leapfrog (CTLF) scheme:

$$\begin{aligned} w^{n+1} &= u^{n-1} + 2\Delta t[\gamma F(v^n) + (1-\gamma)F(w^n)], \\ u^n &= v^n + \frac{\nu\alpha}{2}(w^{n+1} - 2v^n + u^{n-1}), \\ v^{n+1} &= w^{n+1} - \frac{\nu(1-\alpha)}{2}(w^{n+1} - 2v^n + u^{n-1}), \end{aligned}$$

where γ is a real number. A more discriminating filter takes the form $(1, -4, 6, -4, 1)$ instead of $(1, -2, 1)$, and the scheme is

$$\begin{aligned} w^{n+1} &= u^{n-1} + 2\Delta t[\gamma F(v^n) + (1-\gamma)F(w^n)], \\ u^n &= v^n + \nu\alpha(w^{n+1} - 4v^n + 6u^{n-1} - 4u^{n-2} + u^{n-3}), \\ v^{n+1} &= w^{n+1} - \nu(1-\alpha)(w^{n+1} - 4v^n + 6u^{n-1} - 4u^{n-2} + u^{n-3}). \end{aligned}$$

Both methods are computationally more demanding since they require two tendency calculations per time step, which is the most expensive component of contemporary atmosphere and ocean models. Nevertheless, the improvements to the amplitude accuracy are considerable, especially, the latter. The increased accuracy may allow a longer time step for the same error tolerance, tending to offset the increased expense. The RAW-filtered leapfrog scheme is analyzed in [17], and its behavior in IMEX integrations is studied in [2].

Recently, Li and Trenchea [15] proposed a higher-order Robert–Asselin (hoRA) type time filter.

$$\begin{aligned} v^{n+1} &= u^{n-1} + 2\Delta t F(v^n), \\ u^n &= v^n + \frac{\beta}{2}\left(v^{n+1} - 2v^n + u^{n-1}\right) - \frac{\beta}{2}\left(v^n - 2u^{n-1} + u^{n-2}\right), \end{aligned}$$

where the dimensionless parameter $\beta \in [0, 0.4]$. Under the same computational cost as RAW, the hoRA-filtered leapfrog scheme exhibits third-order accuracy. Compared with the third-order Adams-Bashforth method, the hoRA-filtered leapfrog scheme is almost as accurate, stable and efficient, yet easily implementable in legacy codes. A study of the filter in IMEX integrations was conducted in [16].

In the sequel we present the linear analysis for the leapfrog scheme combined with the aforementioned time filters, focusing on the accuracy and stability.

2 Linear analysis of the leapfrog scheme combined with time filters

We now derive the stability condition, amplitude, phase-speed, and the consistency errors. These properties are illustrated by analyzing solutions to the pure oscillation equation (see e.g., [8, 9])

$$\frac{du}{dt} = i\omega\, u, \tag{2.1}$$

where i is the imaginary unit, and ω a real constant. Define the *amplification factor* A as the ratio of the approximate solution at two adjacent time steps, $A = u^{n+1}/u^n$. The amplification factor yields information on two quantities of interest: the amplitude and the relative phase change per time step. Specifically, A can be expressed in modulus-argument form $A = |A|e^{i\theta}$, where

$$|A| = \sqrt{Re(A)^2 + Im(A)^2}, \quad \theta = tan^{-1}(Im(A)/Re(A)).$$

For the true solution to the oscillation equation (2.1), the *exact amplification factor* $A_{\mathrm{e}} = e^{i\omega\Delta t}$ has unity magnitude and phase change $\omega\Delta t$ over a time interval Δt. The amplitude errors are defined as the difference between the magnitude of the approximate amplification factor $|A|$ and the correct value of unity. When $|A| = 1$, the scheme is *neutral*, if $|A| < 1$, the scheme is *damping* (indicating stability), and if $|A| > 1$, it is *amplifying* (instability). The relative phase change or the phase speed, on the other hand, is measured by the ratio of the phase change of the numerical scheme per time step divided by the phase change of the true solution over the same time interval, and is denoted by $R = \theta/\omega\Delta t$. The phase-speed errors are defined as the difference between the phase speed R and the unity over a time interval Δt. When $R > 1$, the method is *accelerating*, and if $R < 1$, it is *decelerating*. Unlike the amplitude, the phase change does not influence the stability of the numerical solution. Instead, the phase errors accumulate and can become large over a long time period of integration.

2.1 The hoRA-filtered leapfrog scheme

The hoRA-filtered leapfrog (LF-hoRA) scheme [15] applied to (2.1) is

$$v^{n+1} = u^{n-1} + 2i\omega\Delta t v^n, \tag{2.2}$$

$$u^n = v^n + \frac{\beta}{2}\left(v^{n+1} - 2v^n + u^{n-1}\right) - \frac{\beta}{2}\left(v^n - 2u^{n-1} + u^{n-2}\right). \qquad (2.3)$$

The system of equations (2.2)-(2.3) is equivalent to the following linear multistep method:

$$u^{n+1} - 2\beta u^n - (1-2\beta)u^{n-1} = i\omega\Delta t(2u^n - 3\beta u^{n-1} + \beta u^{n-2}). \qquad (2.4)$$

2.1.1 Consistency errors, amplitude errors and phase-speed errors

Using Taylor expansion, the local truncation error of (2.4) is shown to be

$$\tau_n(\Delta t) = \frac{2-5\beta}{6}(i\omega\Delta t)^2 u'(t^n) + \frac{11\beta}{12}(i\omega\Delta t)^3 u'(t^n) + \mathcal{O}[(i\omega\Delta t)^4].$$

Thus, the LF-hoRA scheme is second order in general, and third order when $\beta = 0.4$.

Formula (2.4) yields the following equation for the amplification factor:

$$A^3 - 2(\beta + i\omega\Delta t)A^2 + (3\beta i\omega\Delta t - 1 + 2\beta)A - \beta i\omega\Delta t = 0. \qquad (2.5)$$

Equation (2.5) has three roots, one is the physical mode denoted A_p, and the other two are computational modes. Since computational modes are well-controlled by the filter, we focus on the amplitude and phase-speed errors for the physical mode. A series expansion for $|A_p|$ in powers of $\omega\Delta t$ yields the amplitude error as follows:

$$|A_p| - |A_{\mathrm{e}}| = |A_p| - 1 = \frac{\beta(2\beta-3)}{8(1-\beta)^2}(\omega\Delta t)^4 + \mathcal{O}[(\omega\Delta t)^6].$$

The amplitude error after taking a single time step scales as $(\Delta t)^4$, hence it is of order $(\Delta t)^3$ over $T/\Delta t$ time steps. The phase-speed error is

$$R_p - 1 = \frac{\arg(A_p)}{\omega\Delta t} - 1 = \frac{2-5\beta}{12(1-\beta)}(\omega\Delta t)^2 + \mathcal{O}[(\omega\Delta t)^4].$$

The phase speed of the physical mode is fourth-order accurate when $\beta = 0.4$ and second order otherwise.

2.1.2 Stability analysis

To determine the maximum $\omega\Delta t$ for which all numerical amplification factors of the LF-hoRA scheme are non-amplified, we use the *root locus curve* method (see e.g., [11]). The characteristic equation of (2.4) is

$$\zeta^3 - 2\beta\zeta^2 - (1-2\beta)\zeta - z(2\zeta^2 - 3\beta\zeta + \beta) = 0,$$

where ζ denotes the points on the unit circle, i.e., $\zeta = e^{i\theta}$ for $\theta \in [0, 2\pi]$, and $z \in \mathbb{C}$. The curve z is called the *root locus curve.* In our case $z = i\omega\Delta t$ lies on the imaginary axis, and consequently θ satisfies

$$\cos\theta = 1 \text{ or } \cos\theta = \beta - \frac{1}{2}, \quad \text{and hence} \quad z = 0 \text{ or } z = \pm i\frac{\sqrt{\frac{3}{4} + \beta - \beta^2}}{1 + \frac{3}{2}\beta - \beta^2},$$

which indicates the intersections of the root locus curve with the imaginary axis in the complex plane. Thus, the stability of the LF-hoRA scheme is provided by

$$\omega\Delta t \le \frac{\sqrt{\frac{3}{4} + \beta - \beta^2}}{1 + \frac{3}{2}\beta - \beta^2}, \quad 0 < \beta \le 0.4.$$

2.2 The RAW-filtered composite-tendency leapfrog scheme

Notice that the RA-filtered leapfrog (LF-RA) scheme is recovered when $\alpha = 1$ in the RAW-filtered composite-tendency leapfrog (CTLF-RAW) scheme, while LF-RAW scheme is a special case of CTLF-RAW when $\gamma = 1$. For this reason, it suffices to analyze CTLF-RAW (refer to [30] for more details). The scheme applied to (2.1) is

$$w^{n+1} = u^{n-1} + 2i\omega\Delta t(\gamma v^n + (1-\gamma)w^n), \tag{2.6}$$

$$u^n = v^n + \frac{\nu\alpha}{2}(w^{n+1} - 2v^n + u^{n-1}), \tag{2.7}$$

$$v^{n+1} = w^{n+1} - \frac{\nu(1-\alpha)}{2}(w^{n+1} - 2v^n + u^{n-1}). \tag{2.8}$$

The three dimensionless parameters in the scheme are ν, α, and γ, where ν corresponds to the classical Robert–Asselin filter parameter, α partitions the RAW filter displacements between the n'th and $(n+1)$'th time levels, and

γ specifies the weighting coefficients for the composite tendency. Although previous work [30] assumed $0 \leq \gamma \leq 1$, here we allow γ to vary outside this range.

The system of equations (2.6)-(2.8) is equivalent to the following linear multistep method:

$$
\begin{aligned}
&u^{n+1} - \nu u^n - (1-\nu)u^{n-1} \\
&= i\omega\Delta t\Big((2 - \nu\gamma(1-\alpha))u^n + \nu(2\gamma + \alpha - 2 - 2\alpha\gamma)u^{n-1} + \nu(1-\alpha)(1-\gamma)u^{n-2}\Big).
\end{aligned}
\tag{2.9}
$$

2.2.1 Consistency errors, amplitude errors and phase-speed errors

The local truncation error of (2.9) is

$$
\begin{aligned}
\tau_n(\Delta t) = \left(\frac{1}{2} - \alpha\right)\nu(i\omega\Delta t)u'(t^n) + \frac{1}{6}\left(2 - \nu(7 - 9\alpha) + 6\nu\gamma(1-\alpha)\right)(i\omega\Delta t)^2 u'(t^n) \\
+ \frac{\nu}{24}(25 - 28\alpha - 24\gamma + 24\alpha\gamma)(i\omega\Delta t)^3 u'(t^n) + \mathcal{O}(\Delta t^4).
\end{aligned}
$$

The scheme is generally first-order accurate[1] if $\nu \neq 0$, and second order if $\alpha = 0.5$, as noted by Williams [30]. Further, the method becomes third order if $\alpha = 0.5$ and $\gamma = (5\nu - 4)/(6\nu)$. This third-order scheme would require $\gamma < 0$ if $\nu < 4/5$. Finally, the scheme exhibits fourth-order accuracy if $\alpha = 0.5, \nu = -8$, and $\gamma = 11/12$. This case is of no practical interest because the negative value of ν forces the computational mode to be amplified.

Remark 2.1 *The LF-RA scheme is first-order accurate. The LF-RAW is firsr order in general, and second order when $\alpha = 0.5$.*

The amplitude error of CTLF-RAW is given by

$$
|A_p| - 1 = \frac{\nu(1-2\alpha)}{2(2-\nu)}(\omega\Delta t)^2 + \mathcal{O}\left[(\omega\Delta t)^4\right],
$$

yielding first-order amplitude accuracy, independent of γ. Since LF-RA recovers when $\alpha = 1$, its amplitude is therefore first order. When $\alpha = 1/2$, the

[1] If $\nu = 0$, the scheme is generally second order, but then the filter is inactive and the computational mode is uncontrolled. For this reason, $\nu \neq 0$ is not considered throughout the article.

quadratic term vanishes and the amplitude error becomes

$$|A_p| - 1 = \frac{\nu(4\gamma - 3 + \nu - \nu\gamma)}{4(2-\nu)^2}(\omega\Delta t)^4 + \mathcal{O}\left[(\omega\Delta t)^6\right],$$

which implies the third-order amplitude accuracy. The fourth-order term now depends on γ. Specifically, CTLF-RAW is amplifying when $\gamma > (3-\nu)/(4-\nu)$, and is damping if $\gamma < (3-\nu)/(4-\nu)$. Recall that LF-RAW is recovered when $\gamma = 1$, hence it is unstable when $\alpha = 1/2$. When $\gamma = (3-\nu)/(4-\nu)$, the amplitude error is fifth-order accurate:

$$|A_p| - 1 = \frac{\nu}{4(4-\nu)(2-\nu)^2}(\omega\Delta t)^6 + \mathcal{O}\left[(\omega\Delta t)^8\right].$$

However, the coefficient of the sixth-order term is always positive, implying a slight instability of the scheme. The phase-speed error, when $\alpha = 1/2$, is

$$R_p - 1 = \frac{6\nu\gamma + 4 - 5\nu}{12(2-\nu)}(\omega\Delta t)^2 + \mathcal{O}[(\omega\Delta t)^4].$$

The phase speed is fourth-order accurate if further $\gamma = (5\nu - 4)/(6\nu)$.

2.2.2 Stability analysis

Using the similar technique as in Section 2.1.2, we derive the stability condition for the CTLF-RAW scheme. First, the time step condition for LF-RA is

$$\omega\Delta t \le \sqrt{\frac{2-\nu}{2+\nu}}.$$

The LF-RAW is stable under the following condition:

$$\omega\Delta t \le \frac{1}{\alpha}\sqrt{\frac{(2-\nu)(2\alpha-1)}{2-\nu+2\alpha\nu}}, \quad \alpha \in [1/2, 1].$$

For the CTLF-RAW method, it is of more interest when $\alpha = 1/2$ since the scheme is at least second-order accurate. The stability condition in this case is given by

$$\omega\Delta t \le \frac{2}{(1-\gamma)(4-\nu)}\sqrt{\frac{(3-\nu)-(4-\nu)\gamma}{1+\nu(1-\gamma)}}, \quad \gamma \le (3-\nu)/(4-\nu).$$

2.3 The composite-tendency leapfrog scheme with more discriminating filter

Applied to equation (2.1), the scheme [30] is

$$\begin{aligned} w^{n+1} &= u^{n-1} + 2i\omega\Delta t(\gamma v^n + (1-\gamma)w^n), && (2.10)\\ u^n &= v^n + \nu\alpha(w^{n+1} - 4v^n + 6u^{n-1} - 4u^{n-2} + u^{n-3}), && (2.11)\\ v^{n+1} &= w^{n+1} - \nu(1-\alpha)(w^{n+1} - 4v^n + 6u^{n-1} - 4u^{n-2} + u^{n-3}), && (2.12) \end{aligned}$$

which is equivalent to the following linear multistep method:

$$\begin{aligned} &u^{n+1} - \nu(4+3\alpha)u^n - (1-7\nu-\nu\alpha)u^{n-1} - \nu(4-3\alpha)u^{n-2} + \nu(1-\alpha)u^{n-3} \qquad (2.13)\\ &= 2\left(1-\nu(1-\alpha)\gamma\right)u^n - \nu\left(8(1-\alpha)(1-\gamma)+12\alpha\right)u^{n-1}\\ &+ \nu\left(12(1-\alpha)(1-\gamma)+8\alpha\right)u^{n-2} - \nu\left(8(1-\alpha)(1-\gamma)+2\alpha\right)u^{n-3} + 2\nu(1-\alpha)(1-\gamma)u^{n-4}. \end{aligned}$$

2.3.1 Consistency errors, amplitude errors and phase-speed errors

The local truncation error of (2.13) is

$$\begin{aligned} \tau_n(\Delta t) = {} & \frac{1-\nu(1+2\alpha)}{3}(i\omega\Delta t)^2u'(t^n) + \frac{\nu(3-5\alpha)}{3}(i\omega\Delta t)^3u'(t^n)\\ &+ \frac{181\nu - 308\alpha\nu - 120\nu\gamma + 120\alpha\nu\gamma - 1}{60}(i\omega\Delta t)^4u'(t^n) + \mathcal{O}[(i\omega\Delta t)^5]. \end{aligned}$$

Theoretically, the scheme could be third-order accurate if $\alpha = (1-\nu)/(2\nu)$, and even higher-order accurate for appropriate values of the parameters which set zero the coefficients of the higher-order terms. However, the root condition is not satisfied in this case. To see this, set $\omega = 0$ and write (2.13) in terms of the amplification factor A:

$$A^4 - \nu(4+3\alpha)A^3 - (1-7\nu-\nu\alpha)A^2 - \nu(4-3\alpha)A + \nu(1-\alpha) = 0. \quad (2.14)$$

It turns out that when $\alpha = (1-\nu)/(2\nu)$, equation (2.14) has the root $A = 1$ with multiplicity two, violating the root condition (see e.g., [22]). Indeed, the numerical solution grows linearly in time, while is supposed to be constant. Thus, the scheme is second-order accurate.

The amplitude error is

$$|A_p| - 1 = -\frac{\nu(1-2\alpha)}{2(1-\nu-2\alpha\nu)}(\omega\Delta t)^4 + \mathcal{O}[(\omega\Delta t)^6],$$

which, by setting $\alpha = 1/2$, becomes

$$|A_p| - 1 = \frac{\nu(5 - 8\gamma - 9\nu + 14\nu\gamma)}{8(1-2\nu)^2}(\omega\Delta t)^6 + \mathcal{O}[(\omega\Delta t)^8].$$

Further, the sixth-order term vanishes when $\gamma = (5-9\nu)/(2(4-7\nu))$ and gives the seventh-order amplitude error:

$$|A_p| - 1 = -\frac{5\nu(4 - 13\nu + 11\nu^2)}{32(1-2\nu)^2(4-7\nu)}(\omega\Delta t)^8 + \mathcal{O}[(\omega\Delta t)^{10}].$$

The phase-speed error, in this case, is second order:

$$R_p - 1 = \frac{1}{6}(\omega\Delta t)^2 + \mathcal{O}[(\omega\Delta t)^4].$$

2.3.2 Stability

As shown in the consistency error analysis, the composite-tendency leapfrog scheme with the more discriminating filter is second-order accurate regardless of the parameters. Nevertheless, the amplitude exhibits the highest-order of accuracy when $\alpha = 1/2$ and $\gamma = (5-9\nu)/(2(4-7\nu))$. For this reason, we only consider the stability for the chosen values of the parameters. Applying the root locus curve technique, the time step condition for this scheme is

$$\omega\Delta t \leq \sqrt{1 - \left(\frac{8 - 45\nu + 55\nu^2}{12 - 20\nu}\right)^2 \frac{8(4-7\nu)(2-5\nu+5\nu^2)}{(4+25\nu-55\nu^2)(16-68\nu+105\nu^2-55\nu^3)}}. \tag{2.15}$$

3 Conclusions

The development of accurate and efficient time-stepping schemes is an important key in improving the fidelity of the numerical simulations for weather and climate, and is still an active area of research.

We surveyed the recent progress on time filters, a post-processing non-intrusive technique which improves accuracy and stability, and uses legacy codes in a black-box manner. We focus on time filters used in conjunction with the leapfrog scheme, the most commonly employed time-stepping scheme in the weather and climate community. Specifically, we present the

Method	Order	Amplitude	Phase speed	Maximum $\omega\Delta t$
LF-RA	1	$1-\frac{\nu}{2(2-\nu)}p^2$	$1+\frac{1+\nu}{3(2-\nu)}p^2$	$\sqrt{\frac{2-\nu}{2+\nu}}$
LF-RAW	1 or 2	$1-\frac{\nu(2\alpha-1)}{2(2-\nu)}p^2+\mathcal{O}(p^4)$	$1+\left(\frac{(1-\nu(1-\alpha))(2-\alpha\nu)}{(2-\nu)^2}-\frac{1}{3}\right)p^2$	$\frac{1}{\alpha}\sqrt{\frac{(2-\nu)(2\alpha-1)}{2-\nu+2\alpha\nu}}$
LF-hoRA	2 or 3	$1-\frac{\beta(3-2\beta)}{8(1-\beta)^2}p^4$	$1+\frac{2-5\beta}{12(1-\beta)}p^2+\mathcal{O}(p^4)$	$\frac{\sqrt{\frac{3}{4}+\beta-\beta^2}}{1+\frac{3}{2}\beta-\beta^2}$
CTLF-RAW	2 or 3	$1+\frac{\nu(4\gamma-3+\nu-\nu\gamma)}{4(2-\nu)^2}p^4+\mathcal{O}(p^6)$	$1+\frac{6\nu\gamma+4-5\nu}{12(2-\nu)}p^2+\mathcal{O}(p^4)$	$\sqrt{\frac{4((3-\nu)-(4-\nu)\gamma)}{(1+\nu(1-\gamma))[(1-\gamma)(4-\nu)]^2}}$
CTLF-D	2	$1-\frac{5\nu(4-13\nu+11\nu^2)}{32(1-2\nu)^2(4-7\nu)}p^8$	$1+\frac{1}{6}p^2$	Formula (2.15)

Table 1: Comparison between the leapfrog scheme combined with time filters. The amplitude, phase speed, and time step limitations are those associated with the application of each scheme to the oscillation equation (2.1). For brevity, the more discriminating filtered composite-tendency leapfrog scheme is abbreviated by CTLF-D. We denote $p = \omega\Delta t$, and amplitude or phase speed that is with $\mathcal{O}(p^k)$ indicates that it is able to be of order up to p^k.

accuracy and stability analysis of RA, RAW, hoRA, and the more discriminating filtered leapfrog/composite-tendency leapfrog schemes. The properties of these methods are summarized in Table 1, an addendum to the comparison Table 2.2 in [9].

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