Ph.D. PRELIMINARY EXAMINATION

PART I - LINEAR ALGEBRA

MAY 8, 2000

1. Answer at least 4 questions.

2. If you answer more than 4 questions, the best 4 out of 6 will be used to complete your grade.

3. Use a soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.

4. Put your code number, but not your name, on each answer sheet that your submit. Confine your answers to the rectangular area indicated on the answer sheets.

CODE NUMBER: ____________________

GRADE QUESTIONS: 1. ____________ 2. ____________ 3. ____________

4. ____________ 5. ____________ 6. ____________

Preliminary Examination
Linear Algebra

IMPORTANT: While solving the problems below, please justify all your statements by either exhibiting a proof or properly quoting a relevant theorem."

1. Let A and B be real $n \times n$ matrices and let I be the $n \times n$ identity matrix. Prove that $I - BA$ is invertible iff $I - AB$ is invertible and then:

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$

   Hence show that the matrices AB and BA have precisely the same characteristic values.

2. Let P_2 be the vector space of all polynomials $f(x)$, with real coefficients, of degree at most two, in the variable x. Let a, b, c and d be given real numbers. We wish $f(x) \in P_2$ to have the following properties:

$$f(-1) = a, \quad f(1) = b, \quad f(3) = c, \quad f(0) = d.$$

   Prove that $f(x)$ exists iff $3a + 6b - c - 8d = 0$.

3. Let A be a real symmetric $n \times n$-matrix, that is positive definite, so $x^T Ax > 0$ for all $x \neq 0$ in $\boldsymbol{R}^n$. Show that $A_{ii}A_{jj} - A_{ij}^2 > 0$, for all i and j, with $1 \leq i < j \leq n$, where A_{pq}, for any p and q is the (p, q) entry of A.

4. Let P_3 be the vector space of all polynomials $p(x)$, with real coefficients, of degree at most three, in the variable x. Define a map $T : P_3 \to P_3$ by the formula, valid for any $p(x) \in P_3$:

$$(Tp)(x) = \int_0^1 (x - y)^2 p(y) dy.$$

   Prove that T is a linear transformation and find the matrix of T with respect to a standard basis for P_3.
   Find, with proof, bases for the range and the null space of T.

5. Let T be a linear operator on a real vector space $\boldsymbol{V}$ of finite dimension. Suppose that T has rank one. Prove that T is either diagonalizable or nilpotent, but not both.

6. Give, with proof, an example of a 2×2-matrix A, such that A^2 is normal, but A is not normal.