Ph.D. PRELIMINARY EXAMINATION

PART II - ANALYSIS

May 9, 2000

1. Answer at least 4 questions.

2. If you answer more than 4 questions, the best 4 out of 6 will be used to complete your grade.

3. Use a soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.

4. Put your code number, but not your name, on each answer sheet that your submit. Confine your answers to the rectangular area indicated on the answer sheets.

CODE NUMBER: ____________________

GRADE QUESTIONS: 1. ____________ 2. ____________ 3. ____________

4. ____________ 5. ____________ 6. ____________

Preliminary Examination
Analysis

IMPORTANT: While solving the problems below, please justify all your statements by either exhibiting a proof or properly quoting a relevant theorem."

1. Let s be a fixed real number. Define $f : [0, 1] \to \boldsymbol{R}$ by $f(0) = 0$ and if $0 < x \leq 1$, then $f(x) = x^s \sin(\frac{1}{x})$.
   For which values of s is f everywhere bounded?
   For which values of s is f everywhere continuous?
   For which values of s does $f'(0)$ exist?
   For which values of s does $f''(0)$ exist?
   For which values of s does f have infinitely many critical points?
   You may assume standard elementary properties of the trigonometric functions.

2. Let $f_n(x) = \frac{x}{1+nx^2}$, for any real x and any n a positive integer. Prove that $\lim_{n\to\infty} f_n(x)$ converges uniformly to an everywhere differentiable function $f(x)$, but that the sequence $f_n'(x)$ does not converge everywhere to $f'(x)$. For which values of the real number s does the series $g(x) = \Sigma_{n=1}^{\infty}(f_n(x))^s$ converge for all positive x? By finding the maximum value of each $f_n(x)$ or otherwise, prove that the series $g(x)$ converges uniformly if $s > 2$.

3. (a) In a metric space $\boldsymbol{M}$, state at least two equivalent definitions of compactness.

   (b) Prove that the natural numbers $\boldsymbol{N}$, with the metric $d(m, n) = |m - n|$, for all m and n in $\boldsymbol{N}$, is not compact.

   (c) Let x_n be a convergent sequence in a metric space $\boldsymbol{M}$, with limit x. Prove that $\{x_n; n \in \boldsymbol{N}\} \cup \{x\}$ is compact.

   (d) Give an example of a non-convergent sequence of distinct terms, y_n in some metric space, for which $\{y_n; n \in \boldsymbol{N}\}$ is compact.

4. (a) Give the definition of a mapping $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$ being differentiable at a point $a \in \boldsymbol{R}^n$, such that the derivative f' is a linear transformation. Also explain how the first partial derivatives of f are determined by f'.

   (b) Assuming that f has continuous partial derivatives, prove that f is differentiable.

5. Let $f : \boldsymbol{R}^2 \to \boldsymbol{R}^3$ be given by the formula, valid for any $(s, t) \in \boldsymbol{R}^2$:

$$f(s,t) = ((1 + 2\cos s)\cos t, \quad (1 + 2\cos s)\sin t, \quad 2\sin s).$$

   What is the maximum possible value of the rank of the derivative f'? Determine the set S of all points (s, t) in $\boldsymbol{R}^2$ where this maximum value is not attained. Give a rough sketch of the image of f, showing, in particular that it is a surface of revolution and marking the image $f(S)$ of S under f. Show that the complement of the image of f in $\boldsymbol{R}^3$ is the union of three disjoint open sets in $\boldsymbol{R}^3$, A, B and C, say, each of which is connected (to prove connectedness of a region it is enough to prove that any two points of the region may be connected by a continuous path). One of these regions is unbounded, say C. Find the volume of one of the other regions, A or B.

6. Let a be a fixed real number that is not an integer. Let $f(x) = e^{iax}$, defined for any real x. Find the Fourier coefficients of the function f. Hence prove the relations:

$$\Sigma_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}.$$

$$\Sigma_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$