Ph.D. PRELIMINARY EXAMINATION

PART 1 - LINEAR ALGEBRA

January 8, 2003

(1) Answer any 4 of the 6 questions.
(2) Indicate below which 4 questions you wish to have graded. Do not indicate more than 4 questions for grading.
(3) Use a soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.
(4) Put your code number, but not your name, on each answer sheet that you submit. Confine your answers to the rectangular area indicated on the answer sheets.

CODE NUMBER: ____________________

GRADE QUESTIONS: 1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________

(1) Let V be a finite dimensional complex vector space.

(a) Suppose that $U\colon V \mapsto V$ is a linear operator with minimal polynomial

$$(x-\lambda)^r.$$

Use the Jordan form to show the existence of a basis $\mathfrak{B}$ of V such that the matrix of U with respect to $\mathfrak{B}$ is of the form

$$\lambda I + N,$$

where $\lambda \in \mathbb{C}$ and N is a nilpotent matrix.

(b) For U as above prove that

$$\lim_{m\to\infty} U^m v = 0$$

for all $v \in V$ if and only if

$$|\lambda| < 1.$$

(c) Let $T\colon V \mapsto V$ be a linear operator with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, where $k \leq n = \dim(V)$. According to the primary decomposition theorem we can decompose

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$$

where each W_i is a T-invariant subspace and the minimal polynomial of T_i (the restriction of T to W_i) is of the form $(x-\lambda_i)^{r_i}$. Prove that

$$\lim_{m\to\infty} T^m v = 0$$

for all $v \in V$ if and only if

$$|\lambda_i| < 1,$$

for $i = 1, 2, \ldots, k$.

(2) Let A, B, C and D be $n \times n$ complex matrices. Let E be the $2n \times 2n$ matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

(a) Prove that $\det(E) = \det(AD - BC)$ when all the matrices involved A, B, C and D are diagonal.

(b) Prove that $\det(E) = \det(AD - BC)$ when all the matrices involved A, B, C and D are upper-triangular.

(c) Prove that $\det(E) = \det(AD - BC)$ when all the matrices involved A, B, C and D commute.

(3) (a) Let M be a complex matrix with characteristic polynomial $(t-4)^3(t+6)^2$ and minimal polynomial $(t-4)^2(t+6)$. What is its Jordan canonical form?

(b) Let A, B be complex matrices. If the minimal polynomial of A is equal to the minimal polynomial of B and the characteristic polynomial of A is equal to the characteristic polynomial of B, are A and B similar? Prove or give a counterexample.

(c) Suppose that in addition to the assumptions in part 2, we also know that the minimal polynomial of A is equal to the characteristic polynomial of A (and thus that the minimal polynomial of B is equal to the characteristic polynomial of B). Are A and B similar? Prove or give a counterexample.

(4) (a) Show that if O_1, O_2 are orthogonal matrices, then so is O_1O_2.

(b) Show that if λ is an eigenvalue of an orthogonal matrix, then $|\lambda| = 1$.

(5) Let V be a finite dimensional vector space over a field, and let S, T be linear maps from V to V.

(a) Show that

$$\text{rank}(ST) \leq \text{rank}(T)$$

(b) Show that

$$\text{rank}(TS) \leq \text{rank}(T)$$

(c) Show that if S is onto, then

$$\text{rank}(TS) = \text{rank}(ST) = \text{rank}(T)$$

(6) Recall that a real symmetric $n \times n$ matrix M is called positive definite (denoted $M > 0$) if $x^T M x > 0$ for all $x \neq 0$ in $\mathbb{R}^n$. (Here x^T means the transpose of the vector x).

(a) Show that $M > 0$ if and only if all of its eigenvalues are positive.

(b) Show that if $M > 0$, then the largest entry of M is on the diagonal.