Ph.D. PRELIMINARY EXAMINATION

PART II - ANALYSIS

January 10, 2003

Notation: $\mathbb{R}$ denotes the set of all real numbers, while $\mathbb{N}$ denotes the set of all positive integers.

Instructions:

1. Answer Four and only Four Questions out of Six.
2. Indicate below which questions you have answered.
3. Use the answer sheets that have been provided and confine your answers in the rectangular area.
4. Put your code number, but not your name, on each answer sheet that you submit.
5. Write down all your calculations. Explain your proofs and solutions carefully and logically, backing up your assertions by referring to known facts, where appropriate.
6. No notes, books or electronic devices may be used during this exam.

CODE NUMBER: ____________________

GRADE QUESTIONS: 1. ____________ 2. ____________ 3. ____________

4. ____________ 5. ____________ 6. ____________

1. Define the sequence $(a_n)_{n=0}^{\infty}$ of real numbers by:

$$a_0 := 1 \quad ; \text{ and}$$
$$a_n := 1 + \frac{1}{1 + a_{n-1}} \quad ,$$

for all integers $n \geq 1$.

(i) PROVE that $a_n \geq 1$, for all $n \geq 0$.

(ii) PROVE that $(a_n)_{n=0}^{\infty}$ converges to some real number L; and then CALCULATE L.

2. (i) PROVE that there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of distinct positive integers such that

$$\lim_{k \to \infty} \sin(n_k)$$

exists in $\mathbb{R}$.

(ii) Suppose that $A \subseteq \mathbb{R}$ and A is **not** compact. PROVE that there exists a sequence $(F_k)_{k \in \mathbb{N}}$ of closed subsets of $\mathbb{R}$ such that

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots ;$$
$$F_k \cap A \neq \emptyset, \text{ for all } k \in \mathbb{N}, \text{ and}$$
$$A \cap \bigcap_{k \in \mathbb{N}} F_k = \emptyset.$$

[**Hint:** There are two cases to consider.]

3. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of connected subsets of a metric space (M, d).

(i) SHOW that if $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$, then $A := \bigcup_{n\in\mathbb{N}} A_n$ is connected.

(ii) SHOW that if we only assume that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$, then it still follows that $A := \bigcup_{n\in\mathbb{N}} A_n$ is connected.
[**Hint:** Use part (i).]

(iii) Suppose that for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $m \neq n$, such that $A_n \cap A_m \neq \emptyset$. Must we still have that $A := \bigcup_{n\in\mathbb{N}} A_n$ is connected? If so, prove it. It not, give a counterexample.

4. Let $n \in \mathbb{N}$ with $n \geq 2$. Let the function $f : \mathbb{R}^n \to \mathbb{R}$ be such that the partial derivative $\frac{\partial f}{\partial x_i}$ exists at x, for all $x \in \mathbb{R}^n$ and for all $i \in \{1, \ldots, n\}$. Also suppose that $\frac{\partial f}{\partial x_i}$ is a continuous function on $\mathbb{R}^n$, for all $i \in \{1, \ldots, n\}$.

Using the definition of a differentiable function, PROVE that f is differentiable on $\mathbb{R}^n$.

5. In this problem, $||\cdot||$ and $(\cdot, \cdot)$ denote the Euclidean norm and inner product on $\mathbb{R}^n$, respectively. Let the function $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable on $\mathbb{R}^n$. Also assume that $\lim_{||x||\to\infty} f(x) = \infty$. Let us define $m := \inf_{x\in\mathbb{R}^n} f(x)$.

(i) SHOW that every sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ with $\lim_{k\to\infty} f(x_k) = m$ is such that $(x_k)_{k\in\mathbb{N}}$ has a convergent subsequence.

(ii) DEDUCE from (i) that $m > -\infty$ and that there exists $z \in \mathbb{R}^n$ such that $\nabla f(z) = 0$.

(iii) Suppose that $\lim_{||x||\to\infty} \frac{f(x)}{||x||} = \infty$. For each $y \in \mathbb{R}^n$, SHOW that $\lim_{||x||\to\infty} (f(x) - (y, x)) = \infty$ and that there exists $z \in \mathbb{R}^n$ such that $\nabla f(z) = y$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$ such that $f = 0$ outside of $[0, 1]$. For all $x \in \mathbb{R}$, let us define

$$g(x) := \int_0^1 f(t)f(x-t)dt.$$

(i) EXPLAIN why the function g is well defined and continuous on $\mathbb{R}$.

(ii) SHOW that $g = 0$ outside of $[0, 2]$.

(iii) SHOW that, for all $t \in [0, 1]$,

$$\int_0^2 f(x-t)dx = \int_0^1 f(s)ds.$$

(iv) DEDUCE from (iii) that

$$\int_0^2 g(x)dx = \left(\int_0^1 f(t)dt\right)^2.$$