PRELIMINARY EXAM QUESTIONS: LINEAR ALGEBRA

Winter '01-'02

Please solve **six** problems subject to the constraints:

A. Problem 1 **or** Problem 2

B. Problem 3

C. Pick four from problems 4–8.

Please indicate which problems you want graded from the ones you chose!!

Problem 1: Prove (from first principles) that to any linear functional y' on a finite dimensional inner product space V, there corresponds a unique vector y in V such that $y'(x) = (x, y)$ for all x.

Problem 2: Let f be a linear functional on a finite dimensional vector space, V over a field, F, *i.e.* $f : V \to F$.

A: Prove that if there is an $\alpha \in V$ such that $f(\alpha) \neq 0$ then every vector, $\beta \in V$ can be written as

$$\beta = c\alpha + \gamma, \quad \text{where} \quad f(\gamma) = 0 \quad c \in F.$$

B: Let f, g be linear functionals on V and suppose that $f(\gamma) = 0$ implies that $g(\gamma) = 0$. Prove that g is a scalar multiple of f.

Problem 3: True or false; if true, prove it and if false, provide a counterexample. Let $M_n(F)$ denote square matrices over the field F.

a: $A \in M_n(C)$ is orthogonal implies that it is unitary

b: All projections of finite-dimensional vector spaces are diagonalizable.

c: A is normal implies A^{-1} is normal when it exists

d: If the eigenvalues of A lie on the unit circle then A^n is bounded as $n \to \infty$.

e: The minimum polynomial of a real symmetric matrix is a product of distinct linear factors.

CODE NUMBER: ____________________

GRADE QUESTIONS: 1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________ 7. ________ 8. ________

Problem 4: Let A be a real symmetric $n \times n$ matrix. Prove that $\vec{x}^T A\vec{x} > 0$ for all x if and only if all eigenvalues of A are positive.

Problem 5: Prove that the trace of a complex $n \times n$ matrix is the sum of its eigenvalues and that the determinant is the product of the eigenvalues.

Problem 6:

A: Let $A \in M_n(R)$ have a complex eigenvalue, $\alpha + i\beta$ with $\beta \neq 0$ and corresponding eigenvector, $u + iv$. Prove that u and v are linearly independent.

B: Suppose $A \in M_2(R)$ with eigenvalues, $\alpha \pm i\beta$ and eigenvectors, $(u \pm iv)$. Prove there is a matrix, P such that

$$P^{-1}AP = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

C: Find the corresponding matrix P for the matrix

$$A = \begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix}$$

Problem 7: Let $A \in M_n(R)$ have nonegative elements such that the sum of all elements in each row is equal to one. Such a matrix is called a stochastic matrix. Prove the product of two stochastic matrices is a stochastic matrix. Prove that a matrix, A is stochastic if and only if $Ae = e$ where e is the vector

$$e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

Problem 8: Consider the $n \times n$ matrix

$$A = \begin{bmatrix} x+y & x & \cdots & & x \\ x & x+y & & & \vdots \\ \vdots & & \ddots & & x \\ x & x & \cdots & x & x+y \end{bmatrix}$$

A: Prove that $\det(A) = y^{n-1}(nx + y)$. (Hint: Add all columns to the first and then subtract the first row from the others.)

B: Prove that if A^{-1} exists, then it is the same form as A and derive a formula for it.