Ph.D. PRELIMINARY EXAMINATION

PART II – ANALYSIS

January 11, 2002

1. Answer any five of the six problems.
2. Indicate below which questions you have answered.
3. Use the answer sheets that have been provided and confine your answers in the rectangular area.
4. Put your code number, but not your name, on each answer sheet that you submit.

CODE NUMBER: ________________

GRADE QUESTIONS: 1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________

PRELIMINARY EXAM QUESTIONS: ANALYSIS - Winter '01-'02

Notation: Let $]a, b[$ denote the open interval between $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Let log denote the natural logarithm function.

1. Suppose x_n is a sequence in $\mathbb{R}^m$ for some natural number m, with $|x_{n+1} - x_n| \leq 1/(n^2 + n)$ for all n. Prove that this sequence converges.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Define a mapping $g : \mathbb{R}^2 \to \mathbb{R}^2$ by $g(x, y) = (f(x), xf(x) - y)$.

   a) Show that if there exists $x_0 \in \mathbb{R}$ such that $f'(x_0) \neq 0$, then g is invertible near (x_0, y_0) for any $y_0 \in \mathbb{R}$.

   b) Find the inverse of g.

3. Prove or provide a counterexample:

   a) A convergent sequence of real numbers forms a closed subset of $\mathbb{R}$.

   b) Let $f :]0, 1[\to \mathbb{R}$ be uniformly continuous. Then f is bounded.

   c) Let M be a metric space with compact subset $A \subset M$. Suppose a sequence of continuous functions converges pointwise on A, to a continuous limit. Then the convergence is uniform.

4. Define the function

$$f(x) = \int_0^{x^2} e^{-t^2/2}\, dt$$

   Show that $f(x)$ has a power series representation $\sum_{k=0}^{\infty} a_k x^k$ valid for all $x \in \mathbb{R}$, and compute the first two nonzero coefficients.

5. Consider the polynomial function

$$p(x) = x^N + \bar{a}_{N-1}x^{N-1} + \ldots + \bar{a}_1 x + \bar{a}_0$$

   where N is a positive integer, and $\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{N-1}$ are fixed real numbers. Suppose that $x = x_0$ is a simple zero of $p(x)$. Show that if

$$|(a_0, a_1, \ldots, a_{N-1}) - (\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{N-1})|$$

   is sufficiently small, then the function

$$f(x) = x^N + a_{N-1}x^{N-1} + \ldots + a_1 x + a_0$$

   has a zero, say x_1, the value of which depends continuously on $(a_0, a_1, \ldots, a_{N-1})$.

6. For $0 < \alpha \leq 1$, evaluate the following limit:

$$\lim_{n \to \infty} \int_0^1 n \log\left(1 + \left(\frac{x}{n}\right)^{\alpha}\right) dx$$