Ph.D. PRELIMINARY EXAMINATION

PART II - ANALYSIS

January 12, 2001

1. Answer any 4 of the 6 questions.

2. Indicate below which 4 questions you wish to have graded. Do not indicate more than 4 questions for grading.

3. Use of soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.

4. Put your code number, but not your name, on each answer sheet that you submit. Confine your answers to the rectangular area indicated on the answer sheets.

CODE NUMBER: ________________

GRADE QUESTIONS: 1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________

1. Define $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}.$$

Show that f is continuous, and that

$$\lim_{x \to 0^+} f(x) = \infty, \qquad \lim_{x \to \infty} f(x) = 0.$$

2. Let $\{f_n\}$ be a uniformly bounded sequence of functions on $[0,1] \times \mathbb{R}$, and for each n, let $y_n : [0,1] \to \mathbb{R}$ be a solution to the initial value problem

$$\begin{aligned} \frac{dy}{dt} &= f_n(t, y) \\ y(0) &= 0. \end{aligned}$$

Show that the sequence $\{y_n\}$ admits a uniformly convergent subsequence.

3. Let A be the linear transformation of $\mathbb{R}^3$ with matrix

$$\begin{bmatrix} 2 & 5 & -4 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let D be the image under A of the unit ball $\{x \in \mathbb{R}^3 : |x| < 1\}$. Find the volume of D, and justify your answer.

4. Use the Implicit Function Theorem to show that there is no continuously differentiable one to one mapping of $\mathbb{R}^2$ into $\mathbb{R}$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$.

   (a) What does it mean to say that f is differentiable at $p \in \mathbb{R}^2$?

   (b) Give either a proof or a counterexample for the following assertion: If the directional derivatives of f exist in every direction at the point p, then f is differentiable at p.

6. Suppose $g, h \in C^2(\mathbb{R}^3)$, and consider the vector field

$$\mathbf{F} = g \nabla h.$$

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$.

   (a) Use the Divergence Theorem applied to $\mathbf{F}$ to prove that

$$\int_{\Omega} \{g \nabla^2 h + \nabla g \cdot \nabla h\} \, dV = \int_{\partial\Omega} g \nabla h \cdot \mathbf{n} \, dA$$

   where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ and dA is the element of surface area on $\partial\Omega$.

   (b) By making a suitable choice of g, prove that if h is harmonic in Ω (i.e. $\nabla^2 h = 0$ in Ω), and $h = 0$ on $\partial\Omega$, then $h \equiv 0$ in Ω.