Ph.D. PRELIMINARY EXAMINATION

PART I - LINEAR ALGEBRA

January 12, 2000

1. Answer any 4 of the 6 questions.

2. Indicate below which 4 questions you wish to have graded. Do not indicate more than 4 questions for grading.

3. Use a soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.

4. Put your code number, but not your name, on each answer sheet that your submit. Confine your answers to the rectangular area indicated on the answer sheets.

CODE NUMBER: ____________________

GRADE QUESTIONS: 1. ____________ 2. ____________ 3. ____________

4. ____________ 5. ____________ 6. ____________

Linear Algebra, 2000

1. Let V be the vector space of all polynomial functions $p : \mathbb{R} \to \mathbb{R}$ such that $\deg p \leq 2$, so

$$p(x) = c_0 + c_1 x + c_2 x^2.$$

Define three linear functionals f_1, f_2, f_3 on V by

$$f_1(p) = \int_0^1 p(x)dx \quad f_2(p) = \int_0^2 p(x)dx \quad \text{and } f_3(p) = \int_0^{-1} p(x)dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting its dual basis.

2. Let $C \in \mathbb{C}^{n\times n}$ be an $n \times n$ matrix with the block decomposition

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where C_{11} and C_{22} are square and C_{11} is invertible.

(a) Verify for yourself that C can be written as the product

$$C = \begin{pmatrix} I & 0 \\ C_{21}C_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} - C_{21}C_{11}^{-1}C_{12} \end{pmatrix}$$

and deduce from this and from standard properties of determinants (to be spelled out clearly) that

$$\det C = \det C_{11} \ \det(C - C_{21}C_{11}^{-1}C_{12}).$$

Let now $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}$ and $B \in C^{n\times n}$ be given. Denote by $A \otimes B \in \mathbb{C}^{2n\times 2n}$ the $2n \times 2n$ matrix with the block decomposition

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix}.$$

(b) Assume that $a_{11} \neq 0$ and that B is invertible, and deduce from (a) that

$$\det A \otimes B = (\det A \det B)^2.$$

(c) Assume only that B is invertible. Show that the formula $\det A \otimes B = (\det A \det B)^2$ obtained in (b) remains true when $a_{11} = 0$. (**Hint:** Use a continuity argument.)

(d) Show that the formula $\det A \otimes B = (\det A \det B)^2$ obtained in (c) remains true when B is not invertible (and hence is always true). (**Hint:** Use the fact that the spectrum of B consists of a finite number of points and use another continuity argument.)

3. Prove: If V is an inner product space and $\{b_1, b_2, b_3, \cdots, b_n\}$ is a basis for V, then there exists a set $\{a_1, a_2, a_3, \cdots, a_n\}$ of orthogonal vectors in V such that $\{a_1, a_2, a_3, \cdots, a_n\}$ is a basis for V and span $\{b_1, \cdots, b_k\}$ = span $\{a_1, \cdots, a_k\}$ for $k = 1, \cdots, n$.

4. Let F be a field, $F \neq \{0\}$ and $n \geq 1$ an integer.

(a) Let $\xi = (\xi_1, \cdots, \xi_n) \in F^n$ be a nonzero vector. Show that there is an invertible matrix $A \in F^{n\times n}$ such that $Ae^1 = \xi$, where $e^1 = (1, 0, \cdots, 0)$.

(b) Let $\xi = (\xi_1, \cdots, \xi_n) \in F^n$ and $\eta = (\eta_1, \cdots, \eta_n) \in F^n$ be arbitrary *nonzero* vectors. Deduce from (a) that there is an invertible matrix $C \in F^{n\times n}$ such that $C\xi = \eta$.

(c) Let V be an n-dimensional vector space over $F, x \in V$ a given nonzero vector and $\xi = (\xi_1, \cdots, \xi_n) \in F^n, \xi \neq 0$ also given. Deduce from (b) that there is an ordered basis $\{v^1, \cdots, v^n\}$ of V in which the components of x are exactly $(\xi_1, \cdots, \xi_n\}$.

5. Let V be a finite dimensional vector space over a field $F \neq \{0\}$ and T a linear operator on V.

(a) Define "α is a characteristic value of T and a is a characteristic vector".

(b) Prove the following are equivalent for c a scalar in F.

(i) c is a characteristic value of T of T.

(ii) The operator $T - cI$ is singular (not invertible).

(iii) $\det(T - cI) = 0$.

6. Let $A \in \mathbb{C}^{n\times n}$ be a given $n \times n$ matrix over $\mathbb{C}$ and let $K_A = \{B \in \mathbb{C}^{n\times n} : AB = BA\}$.

(a) Show that K_A is a subspace of $\mathbb{C}^{n\times n}$.

(b) Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and let E_λ denote the associated eigenspace of A. Show that if $x \in E_\lambda$ then $Bx \in E_\lambda$ for every $B \in K_A$.

(c) Suppose now that $D \in \mathbb{C}^{n\times n}$ is diagonal with n *distinct* diagonal entries. Deduce from (b) that $B \in K_D$ if and only if B is diagonal.

(d) Suppose that A has n *distinct* eigenvalues. Deduce from (c) that $\dim K_A = n$.