Ph.D. PRELIMINARY EXAMINATION

PART II - ANALYSIS

January 14, 2000

1. Answer any 4 of the 6 questions.

2. Indicate below which 4 questions you wish to have graded. Do not indicate more than 4 questions for grading.

3. Use a soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.

4. Put your code number, but not your name, on each answer sheet that your submit. Confine your answers to the rectangular area indicated on the answer sheets.

CODE NUMBER: ____________________

GRADE QUESTIONS: 1. ____________ 2. ____________ 3. ____________

4. ____________ 5. ____________ 6. ____________

Analysis, 2000

1. Let $\boldsymbol{M}$ be a metric space with metric ρ and let (x_n) be a Cauchy sequence in $\boldsymbol{M}$. Show the following:

(a) For each a in $\boldsymbol{M}$ and each $R > 0$, the ball $B(a : R) = \{x \in \boldsymbol{M} : \rho(x, a) < R\}$ is open in $\boldsymbol{M}$.

(b) The sequence (x_n) is bounded.

(c) If (x_n) has a subsequence (x_{n_k}) which converges to x in $\boldsymbol{M}$, then (x_n) converges to x.

(d) If, in addition, $\boldsymbol{M}$ is compact, then $\boldsymbol{M}$ is complete.

2. Let $a \in \boldsymbol{R}$.

(a) Show that $\lim_{n\to\infty}(1 + \frac{a}{n})^n = e^a$. (Hint: You may use $\ln'(1) = 1$).

(c) Show that if (a_n) is a sequence in $\boldsymbol{R}$, which converges to a, then

$$\lim_{n\to\infty}(1 + \frac{a_n}{n})^n = e^a.$$

(d) Let f be a real-valued differentiable function on $\boldsymbol{R}$ such that the second derivative $f''(0)$ at the origin exists, with $f(0) = f'(0) = 0$ and $f''(0) = a$. Show that

$$\lim_{n\to\infty}(f(\frac{x}{\sqrt{n}}))^n = e^{\frac{ax^2}{2}},$$

for each real $x \in \boldsymbol{R}$.

3. Let

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.$$

(a) Prove that ω is a smooth closed differential two-form for all (x, y, z) in $\boldsymbol{R}^3 - \{(0, 0, 0)\}$.

(b) By integrating over a sphere, or otherwise, show that ω is not exact on its domain. (Hint: $(x, y, z) = (\cos s \sin t, \sin s \sin t, \cos t)$, for suitable ranges of s and t, may serve as a parametrization of the unit sphere.)

4. Use the *Weierstrass Approximation Theorem*, to show that if f is continuous on $[0, 1]$ and $\int_0^1 f(x)x^n dx = 0$, for all non-negative integers n, then f is identically zero on $[0, 1]$.

5. Let (a_n) and (b_n) be two sequences in $\boldsymbol{R}$, and set

$$A_n = \Sigma_{k=0}^{n} a_k, \quad n \geq 0.$$

Show the following:

(a) For each $N \in \boldsymbol{N}$,

$$\Sigma_{n=0}^{N} a_n b_n = \Sigma_{n=0}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N.$$

(b) If (b_n) is decreasing and non-negative, with $\lim b_n = 0$, and if the sequence (A_n) is bounded, then $\Sigma a_n b_n$ converges.

(c) If (b_n) is decreasing and bounded below and if Σa_n converges, then $\Sigma a_n b_n$ converges.

6. Determine the sets of all $z \in \boldsymbol{C}$ in which the following series converge, and find their sums there:

(a)

$$\Sigma_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n.$$

(b)

$$\Sigma_{n=0}^{\infty} (n+1) \left(\frac{z}{1+z}\right)^n.$$