Linear Algebra Preliminary Exam
April 2013

Problem 1 Let $A$ be a $4 \times 4$ real symmetric matrix. Suppose $A^3 + 3A = 0$ and rank $A = 2$, find $\text{tr} A$.

Problem 2 Suppose $A$ and $B$ are square complex matrices, such that $AB$ is a projection. Is it true that $BA$ is a projection? Prove or provide a counterexample.

Problem 3 Let $A$ be a real symmetric positive definite matrix, show that

$$A + A^{-1} - 2I$$

is semi positive definite.

Problem 4 Let $\lambda_i$, $1 \leq i \leq m$ be $m$ different eigenvalues of $A$, show that

$$n(m - 1) \leq \sum_{j=1}^{m} \text{rank} (\lambda_j - A).$$

Problem 5 Suppose $A$ is a complex matrix satisfying $A^3 > 0$. Prove that $A$ is diagonalizable.

Problem 6 Suppose $A$ and $B$ are normal complex $n \times n$ matrices. Prove that

$$sr(AB) \leq sr(A)sr(B).$$

Here $sr(\cdot)$ is the spectral radius of a matrix.