Ph.D. Preliminary Examination (Analysis)

April, 2013

INSTRUCTIONS: *Do all six problems. In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should contain the necessary details. All problems are worth the same number of points.*

1. Let (X, d) be a metric space. Suppose Ω is a compact subset of X and $\mathcal{U}$ is an open cover of Ω. Prove that there exists a $\delta > 0$ such that, for every $\omega \in \Omega$, there exists a $U_\omega \in \mathcal{U}$ such that $B(\omega, \delta) \subseteq U_\omega$.
(Note: Your δ should be independent of ω.)

2. Let f and g be two Riemann integrable functions on $[0,1]$ and $h(x) = \max\{f(x), g(x)\}$ for $x \in [0,1]$.
(i) Prove that h is Riemann integrable on $[0,1]$;
(ii) Suppose that $\{f_n\}$ and $\{g_n\}$ are two sequences of Riemann integrable functions on $[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|dx = \lim_{n\to\infty} \int_0^1 |g_n(x) - g(x)|dx = 0.$$

Let $h_n(x) = \max\{f_n(x), g_n(x)\}$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} \int_0^1 |h_n(x) - h(x)|dx = 0.$$

3. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{x^\alpha}{\sqrt{n}(n^2 + x^3)}$ is uniformly convergent on $[0, \infty)$ if $\alpha = 2$, but not uniformly convergent on $[0, \infty)$ if $\alpha = 3$.

4. Given $u \in C[0,1]$, define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x u(t)dt & \text{if } x \in (0,1], \\ u(0) & \text{if } x = 0. \end{cases}$$

Prove that $\displaystyle\left(\int_0^1 (f(x))^2 dx\right)^{1/2} \le 2\left(\int_0^1 (u(x))^2 dx\right)^{1/2}$.

5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 function such that $\text{rank}(Df(p)) = n$ for every $p \in \mathbb{R}^n$. Let S be a bounded subset of $\mathbb{R}^n$. Prove that, for every $\xi \in \mathbb{R}^m$, $\{x \in S : f(x) = \xi\}$ is a finite set.

6. Let $f, g : \mathbb{R}^3 \backslash \{(0,0,0)\} \to \mathbb{R}$ be two C^2 functions. Let Ω denote the upper hemisphere $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z \ge 0\}$, oriented with unit normal vector $\mathbf{n}$. Let $T = \{(x,y,0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. Prove that

$$\left|\iint_\Omega (\nabla f \times \nabla g) \cdot \mathbf{n} dS\right| \le \pi \sup_T(|f\nabla g| + |g\nabla f|).$$