Linear Algebra Preliminary Examinations, August 2012

1. Let A be a complex square matrix. Suppose that $\operatorname{rank}(A) = \operatorname{tr}(A) = 1$. Prove that A is a projection.

2. Suppose A is a complex nilpotent 2×2 matrix, $AB = -BA$. Prove that either $A = 0$ or $\operatorname{tr}(B) = 0$.

3. Let A be an invertible real symmetric matrix and B a real anti-symmetric matrix such that $AB = BA$. Show that $A + B$ is invertible.

4. Suppose $A + B = C$, where A, B and C are nilpotent matrices. Is it true that A and B commute? Prove or provide a counterexample.

5. Let $A = (a_{ij})_{n \times n}$ be a real symmetric matrix such that $a_{ii} = 0$ for all $1 \leq i \leq n$. Suppose that $I + A$ is positive definite and λ is an eigenvalue of A. Prove that

$$-1 < \lambda < n - 1.$$

6. Suppose A is a complex matrix, such that $e^A = A$. Prove that A is diagonalizable.