Preliminary Exam in Advanced Calculus

August 2012

Problem 1. Let (M, d) be a compact metric space and $z \in M$. Let $T : M \to M$ be a function which satisfies $d(x, y) \leq d(T(x), T(y))$ for all $x, y \in M$, i.e. the distances are non-decreasing under the mapping T. Define $\{x_n\}$ by

$$x_1 = T(z) \quad \text{and } x_{n+1} = T(x_n) \text{ for } n \geq 1.$$

Prove that there exists a subsequence of $\{x_n\}$ which converges to z.

Problem 2. We say that a subset E of a metric space is G_δ if there is a sequence of open sets $\{U_i\}_{i=1}^\infty$ such that $E = \bigcap_{i=1}^\infty U_i$. Let $f : X \to \mathbb{R}$ be a function defined on a metric space. Prove that the set

$$\{x \in X : f \text{ is continuous at } x\}$$

is G_δ. **Hint:** *Let m and n denote positive integers. It is very easy to verify and you can take it for granted that f is continuous at x iff*

$$\forall n\, \exists m\, \forall z, w \left(d(x, z) < m^{-1},\, d(x, w) < m^{-1} \implies |f(z) - f(w)| < n^{-1} \right) .$$

Now the idea is to define sets $U_{n,m}$ and then characterize the set of all points of continuity of f in terms of $U_{n,m}$.

Problem 3. Let $h : [0, 2\pi] \to \mathbb{R}$ be continuous and $D = \{(x, y) : \ x^2 + y^2 \leq 1\}$. Define the sequence of functions $\{F_n\}$ on D by

$$F_n(x, y) = \int_0^{2\pi} \cos(x \sin\theta + y \cos\theta - nh(\theta))d\theta$$

for $(x, y) \in D$ and $n \in \mathbb{N}$. Prove that $\{F_n\}$ has a subsequence that converges uniformly on D.

Problem 4. Let Ω be an open set in $\mathbb{R}^n$. Let $F : \Omega \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}$ be two continuously differentiable functions such that $G \circ F = 0$ on Ω. Suppose that $\displaystyle\sum_{j=1}^n \left(\frac{\partial G(x)}{\partial x_j}\right)^2 > 0$ for every $x \in F(\Omega)$. Prove that $\det(DF) = 0$ on Ω.

Problem 5. Prove that there exists a $\delta > 0$ such that, for each $(a, b) \in \mathbb{R}^2$ satisfying $|(a, b)| = \sqrt{a^2 + b^2} < \delta$, the equation

$$\int_0^1 \left(2te^{t^3(a^3+b^2+x)} - e^{t^2(a^2+b^2-x^2)} \right) dt = 0$$

has a solution x in $(-1, 1)$.

Problem 6. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a continuous function. Let $K \subset \mathbb{R}^3$ be a compact set such that $|f(x) - f(y)| \leq 2012|x - y|^2$ for all $x, y \in K$. Prove that the set $f(K) \subset \mathbb{R}^2$ has measure zero as a subset of $\mathbb{R}^2$.