Linear Algebra Preliminary Examination
August 2010

Problem 1.
Suppose A and B are $n \times n$ matrices, and $A + B$ is invertible. Prove that $\text{rank}A + \text{rank}B \geq n$.

Problem 2.
Suppose A and B are $n \times n$ matrices. Suppose A is normal, B is nilpotent, and $A + B = I$. Prove that $A = I$.

Problem 3.
Let U, V, and W be finite-dimensional subspaces of a linear vector space. Show that

$$\dim(U) + \dim(V) + \dim(W) - \dim(U + V + W) \geq$$
$$\geq \max\{\dim(U \cap V), \dim(V \cap W), \dim(U \cap W)\}$$

Problem 4.
Suppose A and B are 2×2 nilpotent matrices. Prove that AB is diagonalizable.

Problem 5.
Let V be a complex inner product space and T a linear map on V such that $\langle Tx, Ty \rangle = 0$ if $\langle x, y \rangle = 0$. Show that $T = kU$ where U is a unitary map on V.

Problem 6.
Let A and B be two $n \times n$ Hermitian matrices over $\mathbb{C}$ and assume that A is positive definite. Show that all eigenvalues of AB are real.