<u>**Ph.D. Preliminary Examination**</u> (Analysis)

August, 2010

INSTRUCTIONS: *Do all six problems. In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should contain the necessary details. All problems are worth the same number of points.*

1. Let (M, d) be a metric space and $f : M \to M$ be a contraction mapping. Suppose that K is a nonempty compact subset of M which satisfies $f(K) = K$. Prove that K contains exactly one point.

2. For $x = (x_1, x_2) \in \mathbf{R}^2$, let $|x| = (x_1^2 + x_2^2)^{1/2}$. Let $D = \{x \in \mathbf{R}^2 : |x| \leq 1\}$ and $f : D \to \mathbf{R}$ be a continuous function on D. Suppose that $f(x) = f(y)$ whenever $|x| = |y| \leq 1$ and $\int\int_D f(x)|x|^n dA = 0$ for every $n \in \mathbf{N}$. Show that $f(x) = 0$ for every $x \in D$.

3. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a C^1 function on $\mathbf{R}^n$ such that $f(0) = 0$. Let $g = (g_1, \ldots, g_n) : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 function on $\mathbf{R}^n$ such that $g(0) = 0$ and $Dg(0)$ is invertible. Prove that there exists an open neighborhood U of the origin in $\mathbf{R}^n$ and a continuous function $h = (h_1, \ldots, h_n) : U \to \mathbf{R}^n$ such that

$$f(x) = g_1(x)h_1(x) + \cdots + g_n(x)h_n(x)$$

for every $x \in U$.

4. Let $f : [0, 1] \to \mathbf{R}$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$. Suppose that $f(0) < 0 < f(1)$ and $f'(x) \neq 0$ for every $x \in (0, 1)$. Let $S_1 = \{x \in [0, 1] : f(x) > 0\}$ and $S_2 = \{x \in [0, 1] : f(x) < 0\}$. Prove that $\inf(S_1) = \sup(S_2)$.

5. For each $n \in \mathbf{N}$, let $f_n : [0, 1] \to \mathbf{R}$ be an increasing function on $[0, 1]$. Suppose that $\{f_n\}$ converges pointwise to a continuous function f on $[0, 1]$.
 (i) Prove that f is also an increasing function on $[0, 1]$;
 (ii) Prove that $\{f_n\}$ converges uniformly to f on $[0, 1]$.

6. Let ω be the 1-form in $\mathbf{R}^3 \backslash \{(t, t, t) : t \in \mathbf{R}\}$ defined by

$$\omega = \frac{z - y}{(x - z)^2 + (y - z)^2} dx + \frac{x - z}{(x - z)^2 + (y - z)^2} dy + \frac{y - x}{(x - z)^2 + (y - z)^2} dz.$$

Show that ω is closed but not exact.