Linear Algebra Preliminary Examination
August 2009

Problem 1.

Let P be a projector on a finite-dimensional complex vector space V, i.e., a map obeying $P^2 = P$. Without the use of eigenvalues or spectral theory,

a) show that $V = range(P) \oplus null(P)$.

b) show that $range(P) \perp null(P)$ if and only if $P = P^*$.

Problem 2.

Find, with proof, the Jordan canonical form of the matrix A.

$$A = \begin{pmatrix} 2 & 0 & 0 & 2 \\ -1 & 0 & 0 & -2 \\ 5 & 3 & 1 & 10 \\ -1 & 0 & 0 & -1 \end{pmatrix}$$

Problem 3.

Let V be a finite-dimensional space over $\mathbb{C}$. Show that for each linear operator T there exists a basis for V such that T is represented by a triangular matrix in that basis.

NOTE: You are not allowed to use without proof the Jordan canonical form theorem.

Problem 4.

For a complex $n \times n$ matrix A, the matrix e^A is defined as follows.

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{k!}A^k + \ldots$$

a) Prove that e^A is well-defined for any A.

b) Prove that if A is skew-Hermitian (i.e. $\bar{A}^t = -A$), then e^A is unitary.

Problem 5.

Let w be a unit column vector in a finite-dimensional Euclidean vector space, and A be a matrix such that $A = I - 2ww^*$.

a) Show that A is unitary and Hermitian.

b) Show that $null(A - I) = \{w\}^\perp$ and $null(A + I) = span\{w\}$.

c) Find $\det A$.

Problem 6.

Suppose $v_1, v_2, \ldots v_k$ are non-zero vectors is $\mathbb{R}^n$ such that $v_i \cdot v_j \leq 0$ for all $i \neq j$. Determine, with proof, the maximal possible k for $n = 3$, and also for arbitrary n.