Problem 1. Let \((X, d)\) be a metric space and let \(A \subset X\) be a compact subset of \(X\) and assume that \(x \in X \setminus A\). Prove that there exist disjoint open sets \(U \subset X\) and \(V \subset X\) such that \(x \in U\) and \(A \subset V\).

Problem 2. Suppose that \(f : \mathbb{R}^n \to \mathbb{R}^n\) is a Lipschitz function, i.e. \(|f(x) - f(y)| \leq L|x - y|\) for all \(x, y \in \mathbb{R}^n\) and some \(L > 0\) and \(E \subset \mathbb{R}^n\) is a set of measure zero, then \(f(E) \subset \mathbb{R}^n\) has measure zero.

Problem 3. Prove that the two series
\[
\sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad \sum_{n=0}^{\infty} n(n) c_n x^{n+3}
\]
have the same radius of convergence.

Problem 4. Let \(f : [0, 1) \to \mathbb{R}\) be a function that is not necessarily continuous. Define
\[
g(\delta) := \sup\{|f(y) - f(y')| : y, y' \in (1 - \delta, 1)\}.
\]
Prove that \(\lim_{x \to 1^-} f(x)\) exists and is finite if and only if \(\lim_{\delta \to 0^+} g(\delta) = 0\).

Problem 5. Suppose that \(f \in C^2(\mathbb{R}^n \setminus \{0\})\) depends on \(r = |x|\) only, i.e. \(f(x) = g(|x|) = g(r)\) for some \(g \in C^2(0, \infty)\). Express the Laplace operator
\[
\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x)
\]
in terms of \(n, r, g\) and derivatives of \(g\) only.

Problem 6. Prove that if \(K \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})\) satisfies the estimate
\[
|\nabla K(x)| \leq \frac{1}{|x|^3} \quad \text{for all} \ x \neq (0,0)
\]
then there is a constant \(C > 0\) such that
\[
\int_{\{x \in \mathbb{R}^2 : |x| > 2|y|\}} |K(x) - K(x)| \ dx \leq C
\]
for all \(y \in \mathbb{R}^2\).

**Hint:** Use the mean value theorem to estimate \(|K(x) - K(x)|\) and then integrate in polar coordinates.