Preliminary Exam in Advanced Calculus

August 2009

Write solutions to different problems on separate sheets. Use scrap paper first to solve problems and write final solutions in a legible form taking care of clarity of presentation.

Problem 1. Let (X, d) be a metric space and let $A \subset X$ be a compact subset of X and assume that $x \in X \setminus A$. Prove that there exist disjoint open sets $U \subset X$ and $V \subset X$ such that $x \in U$ and $A \subset V$.

Problem 2. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz function, i.e. $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}^n$ and some $L > 0$ and $E \subset \mathbb{R}^n$ is a set of measure zero, then $f(E) \subset \mathbb{R}^n$ has measure zero.

Problem 3. Prove that the two series

$$\sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad \sum_{n=0}^{\infty} n(\log n) c_n x^{n+3}$$

have the same radius of convergence.

Problem 4. Let $f : [0, 1) \to \mathbb{R}$ be a function that is not necessarily continuous. Define

$$g(\delta) := \sup\{|f(y) - f(y')| : \, y, y' \in (1 - \delta, 1)\}.$$

Prove that $\lim_{x \to 1^-} f(x)$ exists and is finite if and only if $\lim_{\delta \to 0^+} g(\delta) = 0$.

Problem 5. Suppose that $f \in C^2(\mathbb{R}^n \setminus \{0\})$ depends on $r = |x|$ only, i.e. $f(x) = g(|x|) = g(r)$ for some $g \in C^2(0, \infty)$. Express the Laplace operator

$$\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x)$$

in terms of n, r, g and derivatives of g only.

Problem 6. Prove that if $K \in C^1(\mathbb{R}^2 \setminus \{(0, 0)\})$ satisfies the estimate

$$|\nabla K(x)| \leq \frac{1}{|x|^3} \quad \text{for all } x \neq (0, 0)$$

then there is a constant $C > 0$ such that

$$\iint_{\{x \in \mathbb{R}^2 : \, |x| > 2|y|\}} |K(x - y) - K(x)| \, dx \leq C$$

for all $y \in \mathbb{R}^2$.
Hint: *Use the mean value theorem to estimate* $|K(x - y) - K(x)|$ *and then integrate in polar coordinates.*