1 Preliminary Exam in Linear Algebra

Fully justify all steps in terms of the major results in linear algebra.

1. Let V be an N dimensional vector space over $\mathbb{R}$. Let

$$\{x_1, x_2, \cdots, x_N\}, \{y_1, y_2, \cdots, y_N\}$$

be two bases. Define a linear operator $A : V \to V$ by

$$Ax_i = y_i, i = 1, \cdots, N.$$

Show that A^{-1} exists.

2. A polynomial $x(t) : \mathbb{C} \to \mathbb{C}$ is even if $x(-t) = x(t)$ and odd if $x(-t) = -x(t)$. Let V be the vector space of all complex valued polynomials and let

$$M := \{x(t) \in V : x \text{ is even }\}, N := \{x(t) : x \text{ is odd }\}.$$

a. Show that M, N are subspaces of V.

b. Show that M and N are each others complements in V, $V = M \oplus N$.

3. Let A be a 3×3 real matrix with characteristic polynomial

$$p(\lambda) = -\lambda^3 - 2\lambda^2 + \lambda + 1.$$

Show $dim(X) \leq 3$ where X denotes the subspace of the vector space of all 3×3 matrices given by

$$X := span\{A^j : j = 0, 1, 2, 3, \cdots\}.$$

4. a. Find a nilpotent N and a scalar λ such that $A = \lambda I + N$ where

$$A = \begin{pmatrix} 7 & \text{-4} \\ 4 & \text{-1} \end{pmatrix}.$$

b. Show that for this A

$$A^{100} = \begin{pmatrix} 3^{100} + 400 \cdot 3^{99} & -400 \cdot 3^{99} \\ 400 \cdot 3^{99} & 3^{100} - 400 \cdot 3^{99} \end{pmatrix}.$$

5. Let V be a finite dimensional inner product space over $\mathbb{R}$. Let $A : V \to V$ be a strictly positive (and thus self adjoint) operator. Show that the functional

$$J(v) := \frac{1}{2}(Av, v) - (b, v), \text{ for any } v \in V,$$

has a unique minimizer and that the minimizer is $x = A^{-1}b$.

6. Let V be a finite dimensional inner product space over $\mathbb{R}$. Let $A : V \to V$ be a strictly positive (and thus self adjoint) operator. For $r > 0$ fixed, define

$$T = I - \frac{1}{r}A.$$

Consider the sequence $x_n \in V$ defined by

$$x_0 \in V \text{ given, and } x_{n+1} = Tx_n .$$

Show that there is an r_0 such that for $r > r_0$

$$x_n \to 0 \text{ as } n \to \infty \text{ for any } x_0 \in V.$$