1 Preliminary Exam in Linear Algebra

Fully justify all steps in terms of the major results in linear algebra.

1. Let $V$ be an $N$ dimensional vector space over $\mathbb{R}$. Let
   \[ \{x_1, x_2, \ldots, x_N\}, \{y_1, y_2, \ldots, y_N\} \]
   be two bases. Define a linear operator $A : V \to V$ by
   \[ Ax_i = y_i, \quad i = 1, \ldots, N. \]
   Show that $A^{-1}$ exists.

2. A polynomial $x(t) : \mathbb{C} \to \mathbb{C}$ is even if $x(-t) = x(t)$ and odd if $x(-t) = -x(t)$. Let $V$ be the vector space of all complex valued polynomials and let
   \[ M := \{x(t) \in V : x \text{ is even}\}, \quad N := \{x(t) : x \text{ is odd}\}. \]
   a. Show that $M, N$ are subspaces of $V$.
   b. Show that $M$ and $N$ are each other’s complements in $V$, $V = M \oplus N$.

3. Let $A$ be a $3 \times 3$ real matrix with characteristic polynomial
   \[ p(\lambda) = -\lambda^3 - 2\lambda^2 + \lambda + 1. \]
   Show $\dim(X) \leq 3$ where $X$ denotes the subspace of the vector space of all $3 \times 3$ matrices given by
   \[ X := \text{span}\{A^j : j = 0, 1, 2, 3, \ldots\}. \]

4. a. Find a nilpotent $N$ and a scalar $\lambda$ such that $A = \lambda I + N$ where
   \[ A = \begin{pmatrix} 7 & -4 \\ 4 & -1 \end{pmatrix}. \]
   b. Show that for this $A$
   \[ A^{100} = \begin{pmatrix} 3^{100} + 400 \cdot 3^{99} & -400 \cdot 3^{99} \\ 400 \cdot 3^{99} & 3^{100} - 400 \cdot 3^{99} \end{pmatrix}. \]

5. Let $V$ be a finite dimensional inner product space over $\mathbb{R}$. Let $A : V \to V$ be a strictly positive (and thus self-adjoint) operator. Show that the functional
   \[ J(v) := \frac{1}{2} (Av, v) - (b, v), \quad \text{for any } v \in V, \]
   has a unique minimizer and that the minimizer is $x = A^{-1}b$.

6. Let $V$ be a finite dimensional inner product space over $\mathbb{R}$. Let $A : V \to V$ be a strictly positive (and thus self-adjoint) operator. For $r > 0$ fixed, define
   \[ T = I - \frac{1}{r} A. \]
   Consider the sequence $x_n \in V$ defined by
   \[ x_0 \in V \text{ given, and } x_{n+1} = Tx_n. \]
   Show that there is an $r_0$ such that for $r > r_0$
   \[ x_n \to 0 \text{ as } n \to \infty \text{ for any } x_0 \in V. \]