Ph.D. Preliminary Examination (Analysis)

August, 2008

1. Suppose f is a real-valued continuous, non-constant function on $[a, b]$, $a < b$. Prove that the range of f is a segment.

2. Suppose that a sequence of functions $f_1(x), f_2(x), \ldots$ converges uniformly on $[0, 1]$ to some function $f(x)$. Suppose further that there exists some constant M such that $|f_i(x)| < M$ for all i and all x. For each n define

$$g_n(x) = \frac{f_1(x) + f_2(x) + \ldots + f_n(x)}{n}.$$

Prove that the sequence of functions $g_1(x), g_2(x), \ldots$ also converges uniformly to the function $f(x)$ on $[0, 1]$.

3. Suppose $f(x)$ is infinitely differentiable on $\mathbf{R}$ and $f(a) = 0$. Prove that $f(x) = (x-a)g(x)$, where $g(x)$ is also infinitely differentiable on $\mathbf{R}$.

4. Let f be a differentiable function on $\mathbf{R}$. Assume that there is no x such that $f(x) = 0 = f'(x)$. Show that the set

$$S = \{x \in [0, 1] : f(x) = 0\}$$

is finite.

5. Let $f : [0, 1] \to \mathbf{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)dx = f(1).$$

6. Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable and f satisfies $f(tx) = t^n f(x)$ for all $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $t \in \mathbf{R}$. Let $f_j = \dfrac{\partial f}{\partial x_j}$. Prove that

$$\sum_{j=1}^{n} x_j f_j(x) = nf(x)$$

for $x \in \mathbf{R}^n$.