Preliminary Exam in Linear Algebra

August 2007

Work each problem on separate sheets of paper. Label each page with your name, problem number, and page number. Justify all steps in terms of the major results in linear algebra.

Problem 1 Show that if Y and Z are subspaces of a finite-dimensional linear space then $\dim Y + \dim Z = \dim(Y+Z) + \dim(Y \cap Z)$

Problem 2: find the Jordan canonical form J of the matrix A below. Also, find the characteristic and minimal polynomials of A, and the matrix T such that $J = T^{-1}AT$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Problem 3: Suppose A is a 2×2 real matrix with trace $tr(A) = 7$ and determinant $\det(A) = 11$. Find $tr(A^2)$.

Problem 4: Let V be a finite-dimensional complex Euclidean space with scalar product $(.,.)$. A linear map T is said to be a *reflection* with respect to a plane $S_u = \{x \in V : (x,u) = 0\}$ if $Tu = -u$ and $Tw = w$ for all w in S_u. Show that T is given by $Tx = x - 2\frac{(x,u)}{(u,u)}u$ and it is an isometry.

Problem 5: Let V be a finitely-dimensional real Euclidean space and let B be a positive semidefinite linear map on V.

(a) Show that $(Bx, x) = 0$ implies $(Bx, y) + (x, By) = 0$ for all y in V.

(Hint: consider the map $t \mapsto (B(x+ty), x+ty), t \in \mathbb{R}$)

(b) Deduce from (a) that the nullspace of B equals the nullspace of B^* and that the nullspace and range of B are orthogonal.

Problem 6: Suppose $v_1, v_2, \ldots, v_{2007}$ are vectors in a 2006-dimensional real Euclidean vector space. Suppose that for all i,j the scalar product (v_i, v_j) is negative. Show that one can express the zero vector as a linear combination of vectors $v_1, v_2, \ldots, v_{2007}$ with positive coefficients.