Preliminary Exam in Advanced Calculus

August 2007

Write solutions to different problems on separate sheets. Use scrap paper first to solve problems and write final solutions in a legible form taking care of clarity of presentation.

Problem 1. Use the definition of a set of measure zero to prove that the graph of a continuous function $f : [0,1] \to \mathbb{R}$ (as a subset of $\mathbb{R}^2$) has measure zero.

Problem 2.
(a) Let $(w_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^k$ such that

$$|w_{n+1} - w_n| < \frac{1}{n^{1.003}}\,, \text{ for all } n \in \mathbb{N}\,.$$

Prove that $(w_n)_{n=1}^{\infty}$ is convergent.
(b) Show that there exists a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{R}$ such that $|z_{n+1} - z_n| = 1/n$, for all $n \in \mathbb{N}$, and yet $(z_n)_{n=1}^{\infty}$ *fails* to converge in $\mathbb{R}$.
(c) Show that there exists a sequence $(y_n)_{n=1}^{\infty}$ in $\mathbb{R}$ such that $|y_{n+1} - y_n| = 1/n$, for all $n \in \mathbb{N}$, and $(y_n)_{n=1}^{\infty}$ converges in $\mathbb{R}$.

Problem 3. Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions from $[0,1]$ into $\mathbb{R}$. Suppose that

- $f_n(x) \geq 0$, for all $x \in [0,1]$, for all $n \in \mathbb{N}$;
- $f_{n+1}(x) \leq f_n(x)$, for all $x \in [0,1]$, and for all $n \in \mathbb{N}$; and
- $\lim_{n\to\infty} f_n(x) = 0$, for all $x \in [0,1]$.

Prove that $(f_n)_{n=1}^{\infty}$ converges uniformly to 0 on $[0,1]$. (*Hint:* If $f_n(x) < \varepsilon$, then f_n is less than ε in an open neighborhood of x; use compactness of $[0,1]$.)

Problem 4. Prove that if $f : [a,b] \to \mathbb{R}$ is continuous, then

$$\lim_{n\to\infty} \sqrt[n]{\int_a^b |f(x)|^n \, dx} = \sup_{x\in[a,b]} |f(x)|.$$

Problem 5. Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives, then f is differentiable.

Problem 6.
(a) State carefully the change of variables formula for a multiple integral.
(b) Let $\varphi_i = \varphi_i(x_1, x_2, \ldots, x_i) : \mathbb{R}^i \to \mathbb{R}$ be C^1 functions for $i = 1, 2\ldots, n-1$. Prove that the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi(x_1, x_2, \ldots, x_n) = (x_1, \varphi_1(x_1) + 2x_2, \varphi_2(x_1, x_2) + 3x_3, \ldots, \varphi_{n-1}(x_1, x_2, \ldots, x_{n-1}) + nx_n)$$

is a diffeomorphism onto an open subset of $\mathbb{R}^n$.
(c) Find the volume of $\Phi((0,1)^n)$, where $(0,1)^n$ is an open unit cube in $\mathbb{R}^n$.