Linear Algebra Preliminary Exam – August 2006

Instructions: Work each problem on a separate sheet of paper. Label the solutions carefully. Fully justify all the steps in terms of the major results in linear algebra.

1. Let V be a vector space and let x, y, z be vectors such that

$$x + y + z = 0$$

Show that $span\{x, y\} = span\{y, z\}$.

2. Let $\langle \cdot, \cdot \rangle$ be non-degenerate, symmetric bilinear form on $\mathbb{R}^n$. We represent all vectors in the standard basis.

a) Show that there exists a symmetric, invertible matrix A such that $\langle x, y \rangle = x^T A y$ for all vectors x and y.

b) Is A^{-1} symmetric ? [Give a proof if so, or present a counterexample if not.]

3. Let V be a finite dimensional complex inner product space and let $A : V \to V$ be Hermitian and positive definite. Using the spectral decomposition theorem, show that for all $x, y \in V$

$$|(Ax, y)| \leq (Ax, x)^{1/2}(Ay, y)^{1/2}.$$

4. Let A be a 3×3 real matrix satisfying $A^3 - 2A^2 + 2A - I_3 = 0$.

a) Find the determinant of A. Justify your answer.

b) Write down an orthogonal matrix with minimal polynomial $X^3 - 2X^2 + 2X - 1$.

5.(a) Let A be a $n \times n$ real symmetric matrix which commutes with all $n \times n$ orthogonal matrices. Show that A must be a scalar multiple of the identity.

(b) Let A be a $n \times n$ orthogonal matrix which commutes with all $n \times n$ orthogonal matrices. Show that A must be a scalar multiple of the identity.

(c) Let A be an invertible $n \times n$ real matrix which commutes with all $n \times n$ orthogonal matrices. Show that A must be a scalar multiple of the identity.

6. Let A be a real $n \times n$ matrix with a positive definite symmetric part $A_s := (A + A^T)/2$. Let

$$r(z) := \frac{1 - z}{1 + z}.$$

Show that all eigenvalues of $r(A)$ lie inside the unit disk in $\mathbb{C}$:

$$|\lambda(r(A))| < 1.$$