Preliminary Exam

August 2006

1. Let ℓ_1 be the set of all infinite sequences $x = (x_1, x_2, \dots, x_k, \dots)$, $x_k \in \mathbb{R}$, such that the series $\sum_{k=1}^{\infty} |x_k|$ converges. Let

$$d(x, y) = \sum_{k=1}^{\infty} |x_k - y_k|$$

define the distance between any two elements x and y in ℓ_1.

(a) Show that $d(x, y)$ is well defined for any x, y in ℓ_1 and prove that (ℓ_1, d) is a metric space.

(b) Prove that (ℓ_1, d) is complete.

2.
(a) Give sufficient conditions under which the series $\sum_{n=1}^{\infty} f_n(x)$ can be differentiated term by term on a bounded interval $I \subset \mathbb{R}$.

(b) Can we differentiate $\sum\limits_{n=1}^{\infty} \arctan \dfrac{x}{n^2}$ term by term on $\mathbb{R}$?

3. Let K be a compact metric space and let $C(K, \mathbb{R})$ denote the metric space of all continuous functions from K into $\mathbb{R}$ with the supremum metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in K\}.$$

Suppose $\mathcal{B} \subseteq C(K, \mathbb{R})$ is such that

(i) if $f, g \in \mathcal{B}$ and $c \in \mathbb{R}$ then $f + g, cf, fg \in \mathcal{B}$;

(ii) the constant function is in $\mathcal{B}$;

(iii) $\mathcal{B}$ for all $x \neq y \in K$ there is an $f \in \mathcal{B}$ such that $f(x) \neq f(y)$.

Then the Stone-Weierstrass Approximation Theorem says that every $f \in C(K, \mathbb{R})$ is the uniform limit of a sequence in $\mathcal{B}$.

(a) Using the Stone-Weierstrass Approximation Theorem show that the set P of all two-variable polynomials is a dense set in $C([0,1]\times[0,1],\mathbb{R})$.

(b) Show that the set $\mathcal{B}$ of all continuous functions $f:\mathbb{R}\to\mathbb{R}$ which are eventually constant (i.e. there are constants c and M so that $f(x)=c$ for all $x\in\mathbb{R}$ such that $|x|\geq M$) satisfies conditions (i), (ii) and (iii) above, but the sine function, $s(x)=\sin(x)$, is not the uniform limit of elements of $\mathcal{B}$.

Why does this not contradict the Stone-Weierstrass Theorem?

4. Let U,V be open subsets of $\mathbb{R}^n$, $\mathbf{a}\in U$ and let $\mathbf{u}\in\mathbb{R}^n$ be a unit vector. Suppose $f:U\to V$ is a continuous bijection such that f and f^{-1} are differentiable.

(a) Define the (total) derivative of f at $\mathbf{a}$.

(b) Define the directional derivative of f at $\mathbf{a}$ in the direction $\mathbf{u}$.

(c) The function $f:\mathbb{R}^2\to(0,\infty)^2$ is given by

$$f(x,y)=(e^x+e^y,e^x+e^{-y}).$$

Calculate the directional derivative of f^{-1} at the point $(2e^c,2\cosh c)$ in the direction $(1/\sqrt{2},1/\sqrt{2})$. State carefully any theorems that you use.

5. Suppose $\phi:[a,b]\to\mathbb{R}$ is differentiable and $f:\phi([a,b])\to\mathbb{R}$ is continuous. Prove the Leibnitz rule:

$$\frac{d}{dx}\left[\int_{\phi(a)}^{\phi(x)}f(t)\,dt\right]=f(\phi(x))\phi'(x).$$

6. Compute

$$\iint_S zdxdy+(5x+y)dydz,$$

where $S=\{(x,y,z)\in\mathbb{R}^3|\ x^2+y^2=z^2,\ 0\leq z\leq 4\}$ with an outward normal.