Linear Algebra Preliminary Exam - August, 2005

Do all four problems:

1. Let H be a complex Hermitian $n \times n$ matrix.

   (a) Show that the eigenvalues of H are real.

   (b) Show that the eigenvectors of H that correspond to distinct eigenvalues are orthogonal.

   (c) Suppose that H is positive definite. Let a be a scalar, w an $(n-1)$-dimensional vector and K an $(n-1) \times (n-1)$ matrix such that

   $$H = \begin{bmatrix} a & w^* \\ w & K \end{bmatrix}$$

   Show that $a > 0$ and that both K and $K-(ww^*)/a$ are positive definite. Show that the eigenvalues of H are positive.

2. (True or False). Give proofs of true statements and 2×2 matrix counterexamples for false statements.

   (a) If all the eigenvalues of A are 0 then $A = 0$.

   (b) Every invertible matrix can be diagonalized.

   (c) If N is a nilpotent matrix with $N^3 = 0$ then $I + N$ has a square root (i.e., there is a square matrix M such that $I + N = M^2$).

   (d) If A commutes with B and B commutes with C then A commutes with C.

   (e) A diagonalizable operator that has only the eigenvalues 0 and 1 is a projection.

   (f) A square matrix A is never similar to $A + I$.

3. Let V be a finite dimensional complex vector space and let $T \in L(V)$. Given $x_0 \in V, x_0 \neq 0$, denote by W the subspace $W = span\{x_0, Tx_0, T^2x_0, \cdots\}$.

   (a) Show that W is T-invariant and that T has at least one eigenvector in W.

   Suppose now that $S \in L(V)$, that $ST = TS$ and that x_0 above is an eigenvector of S associated with the eigenvalue λ.

   (b) Show that $W \subset \ker(S - \lambda I)$.

   (c) Show that T and S have a common eigenvector.

4. Let V and W be finite dimensional vector spaces over the field F. Let $\{a_1, \cdots, a_n\}$ and $\{\beta_1, \cdots, \beta_m\}$ be bases of V and W, respectively.

   (a) Show that the set $\{(\alpha_1, 0), \cdots, (\alpha_n, 0), (0, \beta_1), \cdots, (0, \beta_m)\}$ is a basis of the space $V \times W$.

   (b) Let $\{(\alpha_1, \beta_1), \cdots, (\alpha_{n+m}, \beta_{n+m})\}$ be a basis of $V \times W$. Show that $V = span\{\alpha_1, \cdots, \alpha_{n+m}\}$ and $W = span\{\beta_1, \cdots, \beta_{n+m}\}$.
   Let $T \in L(V)$ and $S \in L(W)$ be given and let $U : V \times W \to V \times W$ be defined by $U(x, y) = (Tx, Sy)$.

   (c) Show that $U \in L(V \times W)$.

   (d) Prove that if T and S are diagonalizable, then U is diagonalizable (use (a)).

   (e) Show that if (α, β) is an eigenvector of U, then either $\alpha = 0$ or α is an eigenvector of T and either $\beta = 0$ or β is an eigenvector of S.

   (f) Prove that if U is diagonalizable, the same thing is true of T and S (use (b) and (e)).