Preliminary Exam in Analysis

August 20, 2005

YOUR NAME (PLEASE PRINT):

1. (10p.) Assume that $E \subset \mathbb{R}^n$ is a compact set and $T : E \to E$ is a continuous mapping such that $|T(x) - T(y)| < |x - y|$ for all $x, y \in E$ with $x \neq y$. Prove that there exists $x_0 \in E$ such that $T(x_0) = x_0$.

2. (10p.) Prove that if the sequence (a_n) of real numbers is convergent to a finite limit $\lim_{n\to\infty} a_n = g$, then

$$\lim_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = g.$$

3. (10p.) Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}$$

is continuous at all real non-integer x i.e. $x \neq \pm n$.

4. (10p.) (a) Complete the definition. *A function* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable at a point* $\mathbf{x} \in \mathbb{R}^n$ *if and only if...*

(b) Prove that the function $f(x, y) = (x^3 + y^3)/\sqrt{x^2 + y^2}$ for $(x, y) \neq (0, 0)$, $f(0, 0) = 0$ is differentiable at $(0, 0)$.

5. (10p.) Let f be a continuous function defined for all real x. Find the derivative $g'(x)$, where

$$g(x) = \int_a^b f(x + t)\, dt.$$

6. (10p.) The usual way to compute the derivative of the function $f(x) = x^x$ is to write the function in the form $y = e^{x \ln x}$ and then it is easy. *Do not* use this

method, but find the derivative of the function $f(x) = x^x$ using the following argument: represent the function as a composition of $g(x, y) = x^y$ with the function $h(x) = (x, x)$, $f(x) = (g \circ h)(x)$ and then apply the chain rule to find the derivative. Show all your work.

7. (10p.) Use the transformation $u = x + 2y$, $v = x - y$ to evaluate the integral

$$\int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{(y-x)}\, dx\, dy$$

by first writing it as an integral over a region G in the uv-plane. (Your solution *must* use the above change of variables. There is no partial credit for a direct evaluation of the integral.)

8. (10p.) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Prove that there exists a continuous function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x) - f(y) = (x - y) \cdot g(x, y).$$

Here $\cdot$ denotes the dot product of vectors in $\mathbb{R}^n$. (Once you find a formula for $g(x, y)$, a substantial part of the problem is the proof of the continuity of g.)

9. (10p.) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (u(x, y), v(x, y))$ be a mapping of class C^2 such that partial derivatives of u and v are bounded on $\mathbb{R}^2$ and $|f(x, y)| \leq 1/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. Prove that

$$\iint_{\mathbb{R}^2} \frac{\partial(u, v)}{\partial(x, y)}\, dx\, dy = 0.$$

Here the integral over $\mathbb{R}^2$ is understood as the improper integral $\iint_{\mathbb{R}^2}(\ldots) = \lim_{r\to\infty} \iint_{D(0,r)}(\ldots)$, where $D(0, r)$ denotes the disc of radius r centered at the origin.
(Hint: represent the Jacobian $\partial(u, v)/\partial(x, y)$ as a divergence of some vector field.)