LINEAR ALGEBRA PRELIMINARY EXAM, August, 2004

In the following, n, m are generic positive integers, $\mathbb{F}$ a generic field, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers, $M_{m,n}(R)$ the set of all $m \times n$ matrices with entries in a ring R, $M_n(R) = M_{n,n}(R)$, I the identity matrix of appropriate rank, and I the identity map.

1. (10pts) Let $A = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$. Calculate A^{100}. (Hint: $A = I + N$ where $N^2 = 0I$.)

2. (10pts) Find a linear change of variables from $(x, y) \in \mathbb{R}^2$ to $(z, w) = (ax + by, cx + dy)$ such that for all $(x, y) \in \mathbb{R}^2$,

$$x^2 + y^2 = z^2 + w^2, \qquad x^2 + 2xy - 3y^2 = \lambda z^2 + \mu w^2$$

where λ, μ are some real numbers.

3. (10pts) Suppose $A \in M_n(\mathbb{R}), B \in M_{n,m}(\mathbb{R}), C \in M_{m,n}(\mathbb{R}), D \in M_{m,m}(\mathbb{R})$, and the determinant $\det(A)$ of A is non-zero. Show that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B).$$

(Hint: Multiply the matrix by an appropriate triangular matrix.).

4. (10pts) Let V be a vector space over a field $\mathbb{F}$ and $f, g : V \to \mathbb{F}$ be two linear functionals. Assume that

$$\{\mathrm{x} \in V \mid f(\mathrm{x}) = 0\} = \{\mathrm{y} \in V \mid g(\mathrm{y}) = 0\}.$$

Show that there exists a non-zero constant c such that $f = cg$.

5. (10 pts) <u>*Choose any one of the following two problems.*</u>

   (a) Find integers a, b, c, d such that the following two matrices are similar over $\mathbb{Q}$:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & d \end{pmatrix}.$$

   (b) (10pts) Find a matrix, A, of <u>integer</u> entries such that $A \neq I$ and $A^3 = I$.

6. (10 pts) <u>*Choose any one of the following two problems.*</u>

   (a) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be defined by $TB = AB - BA$. Find a basis for the kernel $\ker(T)$ of T and a basis for the range $R(T)$ of T.

(b) Let E_2, a vector space over $\mathbb{R}$, be the set of all polynomials of real variable t, with real coefficients, and of degree ≤ 2. Let $\mathrm{T} : E_2 \to E_2$ be defined by

$$\mathrm{T}p(t) = t^2 p''(t) + tp'(0) + \int_0^1 p(s)\,ds \qquad \forall\, p \in E_2.$$

Find all the eigenvalues, including their multiplicity, of T.

7. (20 pts) <u>*Choose any one of the following two problems.*</u>

(a) Let $A \in M_n(\mathbb{R})$ be symmetric positive definite and $b \in \mathbb{R}^n$. Define

$$\phi(x) = x^T A x - 2b^T x \qquad \forall x \in \mathbb{R}^n.$$

Show that $z \in \mathbb{R}^n$ solves $Az = b$ if and only if $\phi(z) \leq \phi(x)$ for all $x \in \mathbb{R}^n$.

(b) Let $C([0,1])$ be the Euclidean space of all continuous functions defined on $[0,1]$ equipped with the inner product $\langle p, q\rangle = \int_0^1 p(t)q(t)\,dt$. Let E_n be the Euclidean subspace consisting of all polynomials of real variable t, with real coefficients, and of degree $\leq n$ ($n \geq 1$ an integer). Prove the following:

(i) $\forall\, f \in C([0,1])$, there is a unique $f_n \in E_n$ such that $\langle f, q\rangle = \langle f_n, q\rangle \quad \forall q \in E_n$;

(ii) the map $\mathbf{P}_n : f \in C([0,1]) \to f_n \in E_n$ is an orthogonal projection.

8. (20 pts) <u>*Choose any one of the following two problems.*</u>

You can use the fact that *any linear map of a finite dimensional non-trivial vector space over $\mathbb{C}$ to itself admits at least an eigenpair.*

(a) Let E be a finite dimensional Euclidean space over $\mathbb{C}$ and $\mathbf{A} : E \to E$ be a linear operator. Suppose that

$$\forall\, \lambda \in \mathbb{C}, \qquad \ker(\lambda\mathbf{I} - \mathbf{A}) \perp R(\lambda\mathbf{I} - \mathbf{A}),$$

where ker and R denote the kernel and range respectively. Show that there is an orthonormal basis of E consisting of eigenvectors of $\mathbf{A}$.

(b) Let V be a finite dimensional non-trivial vector space over $\mathbb{C}$ and $\mathbf{A}_1, \cdots, \mathbf{A}_m$ be linear operators from V to V. Assume that $\mathbf{A}_i\mathbf{A}_j = \mathbf{A}_j\mathbf{A}_i$ for all $i, j = 1, \cdots, m$. Show that $\mathbf{A}_1, \cdots, \mathbf{A}_m$ share at least one common eigenvector, i.e. there exists $\mathbf{x} \in V$, $\mathbf{x} \neq 0$, such that $\mathbf{A}_i\mathbf{x} = \lambda_i\mathbf{x}$ for all $i = 1, \cdots, m$ where $\lambda_1, \cdots, \lambda_m \in \mathbb{C}$.