PRELIMINARY EXAM IN ANALYSIS August, 2004

You have 180 minutes to complete this exam. Each problem is worth 20 points. Be sure to justify your reasoning and to complete all parts of all problems. In all problems with multiple parts, you may use earlier parts in the answers of later parts, even if you cannot manage to complete the earlier parts.

1. For all parts of this problem, let $K \subset \mathbb{R}$ be a compact set and let $\{f_n\}$ be a sequence of continuous functions on K.

   a) Give an example to show that even if $f_n \to f$ on K, f does not have to be continuous.

   b) Give an example to show that the space $\mathcal{C}(K, \mathbb{R})$ is not complete under the norm $||f|| = \int_K |f|\,dx$.

   c) Prove that if $f_n \to f$ uniformly on K, then f is uniformly continuous on K.

2. a) Prove that if a set $A \subset \mathbb{R}^n$ is both open and closed, then bd(A) is the empty set.

   b) Suppose that $A \subset \mathbb{R}^n$ is bounded and nonempty, with $x \in A$ and $y \in \mathbb{R}^n \setminus A$. Let $\phi(t)$ denote the line segment connecting x to y, parametrized by $t \in [0,1]$, such that $\phi(0) = x$ and $\phi(1) = y$. Prove that there exists $t^* \in [0,1]$ such that $\phi(t^*) \in$bd(A).

   c) Use parts a) and b) to prove that if a set $A \subset \mathbb{R}^n$ is both open and closed, then A is either the empty set or is $\mathbb{R}^n$ itself.

3. a) Prove that if $f : [a,b] \to \mathbb{R}$ is bounded and Riemann integrable, then $|f| : [a,b] \to \mathbb{R}$ is also Riemann integrable.

   b) Suppose $f : [0,1] \to \mathbb{R}$ is a bounded, integrable function such that for all $a, b \in [0,1]$, there exists $c \in (a,b)$ such that $f(c) = 0$. Prove that $\int_0^1 f = 0$.

4. Check the pointwise and uniform convergence of the following series

$$\sum_{n=1}^{\infty} \frac{x^2}{n(n+x^2)}$$

   a) on $(0,1)$.

   b) on $(1,\infty)$.

5. Determine the set of all values of α for which the function

$$f(x,y) = \begin{cases} \dfrac{|x|^{3\alpha} + |y|^{7-\alpha}}{x^2+y^2}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0 \end{cases}$$

   is differentiable at $(0,0)$.

6. a) Prove that

$$\int_0^\infty \left(\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nx} \right) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

b) Use part a) to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2).$$