Ph.D. PRELIMINARY EXAMINATION

PART I – LINEAR ALGEBRA

August 21, 2003

1. Answer 3 problems from Part 1 and 3 problems from Part 2. Do not do more than 6 problems. The total time for the exam is three hours.

2. Indicate below which problems you wish to have graded.

3. Use of soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.

4. Put your name on each answer sheet that you submit.
Confine your answers to the rectangular area indicated on the answer sheets.

NAME: ______________________________

PART 1
GRADE PROBLEMS: 1. ________ 2. ________ 3. ________

4. ________

PART 2
GRADE PROBLEMS: 1. ________ 2. ________ 3. ________

4. ________

August 2003 Linear Algebra prelim

Instructions: Do 3 problems from part 1 and 3 problems from part 2. Each problem is worth 20 points. Do not do more than 6 questions.

Part 1: Do 3 out of 4.

1. Let $M_n(R)$ denote the vector space of real $n \times n$ matrices. Let $T : M_n(R) \to M_n(R)$ be defined by $T(A) = A^t$, the transpose of the matrix A. Prove that T has only two eigenvalues, and that their eigenvectors span $M_n(R)$.

2. (a), Find a scalar λ and a nilpotent matrix N such that

$$\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + N.$$

(b). Use part (a) to show that if P is a polynomial, then

$$P\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} P(2) + P'(2) & P'(2) \\ -P'(2) & P(2) - P'(2) \end{bmatrix}.$$

3. Let $P_n(x)$ denote the vector space of polynomials with real coefficients and of degree less than or equal to n. Show that the linear transformation given by differentiation:

$$D : p \to p' \tag{1}$$

is not diagonalizable. That is, any matrix which is the matrix for this transformation relative to a given basis for P_n is not diagonalizable by a similarity transformation.

4. Let V be a finite dimensional vector space over the real numbers with an inner product $< \mathrm{v}, \mathbf{w} >$. Let f be a linear functional on V, that is, a linear transformation $f : V \to R$. Prove that there exists a unique vector $\mathrm{v}_0 \in V$ such that $f(\mathrm{v}) = < \mathrm{v}, \mathrm{v}_0 >$ for all $v \in V$.

Part 2: Do 3 out of 4.

5. Let A be a linear map on an n-dimensional complex vector space. Let $\lambda_1, \cdots, \lambda_n$ be its eigenvalues. Show that

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq trace(A^* A)$$

and that equality holds if and only if A is normal.

6. (a). If A is Hermitian, show that its largest eigenvalue is $max_{x^* x=1} x^* Ax$.

(b). If A is Hermitian and has at least one positive eigenalue, show that

$$max_{x^* x=1} x^* Ax = max_{x^* Ax=1} \frac{1}{x^* x}.$$

7. Consider the space of polynomials of degree $\leq n$ in the variable t and with complex coefficients. Give this space the inner product $(x, y) = \int_0^1 x(t)\overline{y(t)}dt$.

(a). Is the multiplication operator T which acts by $Tx(t) = t * x(t)$ self adjoint?

(b). Is the differentiation operator D self adjoint?

To receive credit you must prove that your answers are correct.

8. (a). Let O be a $2 * 2$ real orthogonal matrix with determinant 1. Prove that it corresponds to rotation by some angle in the Euclidean plane.

(b). Let O be a $3 * 3$ real orthogonal matrix with determinant 1. Prove that O has 1 as an eigenvalue.

(c). Let O be a $3 * 3$ real orthogonal matrix with determinant 1. Show that O represents rotation about some line in 3 space.