**Preliminary Exam in ANalysis August 2003**

Solve any 4 out of the 6 problems.

**1.** Let $(x_n)$ be a sequence of real numbers. Define the sequence $(y_n)$ by

$$y_n = \frac{1}{n}\sum_{j=1}^{n} x_j.$$

**(a)** Prove that if $(x_n)$ converges to $x \in R$, then $(y_n)$ converges to $x$ also.

**(b)** Construct a nonconvergent sequence $(x_n)$ such that $(y_n)$ is convergent to some $x \in R$.

**2.** Let $f : R^2 \longrightarrow R$ be a twice continuously differentiable function satisfying

$$f(0,y) = 0, \text{ for all } y \in R.$$

**(a)** Show that $f(x,y) = xg(x,y)$ for all pairs $(x,y) \in R^2$, where $g$ is the function given by

$$g(x,y) = \int_0^1 \frac{\partial f}{\partial x}(tx,y)dt.$$

**(b)** Show that $g$ is continuously differentiable and that, for all $x \in R$,

$$g(0,y) = \frac{\partial f}{\partial x}(0,y), \ \frac{\partial g}{\partial y}(0,y) = \frac{\partial^2 f}{\partial x \partial y}(0,y).$$

**(c)** Deduce from (a) and (b) that:

- If $\frac{\partial f}{\partial x}(0,0) \neq 0$, there is a neighborhood $V$ of (0,0) in $R^2$ such that $f^{-1}(0) \cap V = V \cap \{x = 0\}$.
- If $\frac{\partial f}{\partial x}(0,0) = 0$, and $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq 0$, there is a neighborhood $V$ of (0,0) in $R^2$ such that $f^{-1}(0) \cap V$ consists of the union of the set $V \cap \{x = 0\}$ with a curve through (0,0) whose tangent at (0,0) is not vertical (that is, not parallel to the $y-$axis).

**3.** Let $A \subseteq R^n$ be a set that is not compact. Show that there exists a sequence of closed sets $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k \supseteq \cdots$ such that $F_j \cap A$ is nonempty for all $j$, and $(\cap_k F_k) \cap A$ is empty.

**4.** Let $X$ denote the vector space of real-valued continuous functions on [0,1], and let $||\cdot||_\infty$ and $||\cdot||_1$ be two norms on $X$ defined by

$$||x||_\infty = max_{t\in[0,1]}|x(t)|, \ ||x||_1 = \int_0^1 |x(t)|dt.$$

**(a)** Why are $||x||_\infty$ and $||x||_1$ well defined for every $x \in X$? (You are not asked to prove that they are norms, only to justify their definition.)

**(b)** Show that if $(x_n) \subseteq X$ and there is $x \in X$ such that $lim_{n\to\infty}||x_n - x||_\infty = 0$, then $lim_{n\to\infty}||x_n - x||_1 = 0$. Provide a counterexample showing that the converse is not true.

**(c)** For $k \in N$, let $P_k$ denote the space of real polynomials of degree $\leq k$. Show that if $(x_n) \subseteq P_k$ and there is $x \in X$ such that $lim_{n\to\infty}||x_n - x||_1 = 0$, then $x \in P_k$ and $lim_{n\to\infty}||x_n - x||_\infty = 0$.

**5.** Let $f_n : [0,1] \to R$ be continuous functions such that $f_n \to f$ uniformly on $[0,1]$. Let $0 < x_n \leq 1$ be such that $x_n \to 1$.

**(a)** Show that $\int_0^{x_n} f_n(x)dx \to \int_0^1 f(x)dx$.

**(b)** Is part (a) true if the convergence $f_n \to f$ is not uniform? If the answer is **yes** state the appropriate result, if the answer is **no** give a counterexample.

**6.** Imagine the surface of the Earth as a sphere. Temperature on the surface of the Earth is a continuous function. Intersect the surface of the Earth with a plane such that the resulting curve is a circle $C$ of positive radius. Prove that there exist two diametrically opposed points on $C$ having the same temperature.