Preliminary Exam in Analysis August 2003

Solve any 4 out of the 6 problems.

1. Let \((x_n)\) be a sequence of real numbers. Define the sequence \((y_n)\) by

\[ y_n = \frac{1}{n} \sum_{j=1}^{n} x_j. \]

(a) Prove that if \((x_n)\) converges to \(x \in R\), then \((y_n)\) converges to \(x\) also.

(b) Construct a nonconvergent sequence \((x_n)\) such that \((y_n)\) is convergent to some \(x \in R\).

2. Let \(f : R^2 \rightarrow R\) be a twice continuously differentiable function satisfying

\[ f(0, y) = 0, \text{ for all } y \in R. \]

(a) Show that \(f(x, y) = xg(x, y)\) for all pairs \((x, y) \in R^2\), where \(g\) is the function given by

\[ g(x, y) = \int_0^1 \frac{\partial f}{\partial x}(tx, y) dt. \]

(b) Show that \(g\) is continuously differentiable and that, for all \(x \in R\),

\[ g(0, y) = \frac{\partial f}{\partial x}(0, y), \quad \frac{\partial g}{\partial y}(0, y) = \frac{\partial^2 f}{\partial x \partial y}(0, y). \]

(c) Deduce from (a) and (b) that:

- If \(\frac{\partial f}{\partial x}(0, 0) \neq 0\), there is a neighborhood \(V\) of \((0,0)\) in \(R^2\) such that \(f^{-1}(0) \cap V = V \cap \{x = 0\}\).

- If \(\frac{\partial f}{\partial x}(0, 0) = 0\), and \(\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq 0\), there is a neighborhood \(V\) of \((0,0)\) in \(R^2\) such that \(f^{-1}(0) \cap V\) consists of the union of the set \(V \cap \{x = 0\}\) with a curve through \((0,0)\) whose tangent at \((0,0)\) is not vertical (that is, not parallel to the \(y\)-axis).

3. Let \(A \subseteq R^n\) be a set that is not compact. Show that there exists a sequence of closed sets \(F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k \supseteq \cdots\) such that \(F_j \cap A\) is nonempty for all \(j\), and \((\cap_k F_k) \cap A\) is empty.
4. Let $X$ denote the vector space of real-valued continuous functions on $[0,1]$, and let $\| \cdot \|_\infty$ and $\| \cdot \|_1$ be two norms on $X$ defined by

$$\|x\|_\infty = \max_{t \in [0,1]} |x(t)|, \quad \|x\|_1 = \int_0^1 |x(t)| \, dt.$$ 

(a) Why are $\|x\|_\infty$ and $\|x\|_1$ well defined for every $x \in X$? (You are not asked to prove that they are norms, only to justify their definition.)

(b) Show that if $(x_n) \subseteq X$ and there is $x \in X$ such that $\lim_{n \to \infty} \|x_n - x\|_\infty = 0$, then $\lim_{n \to \infty} \|x_n - x\|_1 = 0$. Provide a counterexample showing that the converse is not true.

(c) For $k \in \mathbb{N}$, let $P_k$ denote the space of real polynomials of degree $\leq k$. Show that if $(x_n) \subseteq P_k$ and there is $x \in X$ such that $\lim_{n \to \infty} \|x_n - x\|_1 = 0$, then $x \in P_k$ and $\lim_{n \to \infty} \|x_n - x\|_\infty = 0$.

5. Let $f_n : [0,1] \to \mathbb{R}$ be continuous functions such that $f_n \to f$ uniformly on $[0,1]$. Let $0 < x_n \leq 1$ be such that $x_n \to 1$.

(a) Show that $\int_0^{x_n} f_n(x) \, dx \to \int_0^1 f(x) \, dx$.

(b) Is part (a) true if the convergence $f_n \to f$ is not uniform? If the answer is yes state the appropriate result, if the answer is no give a counterexample.

6. Imagine the surface of the Earth as a sphere. Temperature on the surface of the Earth is a continuous function. Intersect the surface of the Earth with a plane such that the resulting curve is a circle $C$ of positive radius. Prove that there exist two diametrically opposed points on $C$ having the same temperature.