Linear Algebra Preliminary Examinations, April 2012

1. Let V be a finite dimensional vector space over a field F. Let $A, B : V \to V$ be two linear transformations. Prove that

$$\dim (\ker (AB)) \geq \max \{\dim (\ker A), \dim (\ker B)\}.$$

2. Let A and B be two complex square matrices. Suppose $AB - I$ is a projection. Is it true that $BA - I$ is a projection? Prove or provide a counterexample.

3. Suppose A is a complex nilpotent 2×2 matrix, $AB = -BA$. Is it true that $AB = 0$? Prove or provide a counterexample.

4. Suppose λ_1 is an eigenvalue of a complex matrix $A_{n \times n}$ of algebraic multiplicity k. Show that the rank of $(A - \lambda_1 I)^k$ is $n - k$.

5. Suppose A and B are complex matrices, $AB - BA$ is positive semi-definite. Prove that A and B have a common eigenvector.

6. Let A, B be two $n \times n$ positive definite matrices. Show that

$$\det (A + B) \geq \det (A) + \det (B).$$