Preliminary Exam in Advanced Calculus

April 2012

Write solutions to different problems on separate sheets. Use scrap paper first to solve problems and write final solutions in a legible form taking care of clarity of presentation.

Problem 1. Let $\{x_n\}_{n=1}^{\infty}$be a nonnegative, decreasing sequence such that $\sum_{n=1}^{\infty} x_n$ is convergent. Prove that $\sum_{n=1}^{\infty} 2^n x_{2^n}$ is also convergent.

Problem 2. Let x_{ij}, $i,j \in \mathbb{N} = \{1,2,\ldots\}$ be real numbers bounded by 1, $|x_{ij}| \leq 1$ for all $i,j \in \mathbb{N}$. Prove that there is a sequence $j_i < j_2 < j_3 < \ldots$ such that for every $i \in \mathbb{N}$, the sequence $(x_{ij_n})_{n=1}^{\infty}$ is convergent.

Problem 3. Suppose that $f : (0,1) \to \mathbb{R}$ is continuous and bounded and such that the limit $\lim_{x\to 0^+} f(x)$ does not exist. Prove that there is an interval $[a,b]$, $a < b$ such that for every $x \in [a,b]$ there is a decreasing sequence $x_k \in (0,1)$, $x_k \to 0$ such that $f(x_k) \to x$.

Problem 4. Given a positive integer n, prove that there is $\varepsilon > 0$ such that for every $n \times n$ matrix A with $|A| < \varepsilon$ (Hilbert-Schmidt norm), there is an $n \times n$ matrix B such that $A = B^2 + B$. **Hint:** *Differentiate* $f(B) = B^2 + B$.

Problem 5. For $a = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$, $a_n \neq 0$ let $P_a(x) = a_n x^n + \ldots + a_1 x + a_0$. Suppose that for $a^0 = (a_0^0, \ldots, a_n^0)$, $a_n^0 \neq 0$ the polynomial $P_{a^0}(x)$ has n distinct real roots. Prove that there is $\varepsilon > 0$ and C^∞ smooth functions

$$\lambda_1, \ldots, \lambda_n : B^{n+1}(a^0, \varepsilon) \to \mathbb{R}$$

such that for any $a \in B^{n+1}(a^0,\varepsilon)$, $\lambda_1(a), \ldots, \lambda_n(a)$ are distinct roots of the polynomial $P_a(x)$. In other words, prove that in a small neighborhood of a^0, roots of the polynomial P_a depend smoothly on the coefficients $a_0, a_1, \ldots, a_n$.

Problem 6. Let $x = (x_1, x_2)$ and $f(x)$ be a C^2 function on $\mathbb{R}^2$ such that $|\partial^2 f(x)/\partial x_j \partial x_k| \leq 1$ for all $x \in \mathbb{R}^2$ and $j,k \in \{1,2\}$. Prove that

$$\left| \iint_{|x|\leq 1} f(x)dA - \pi f(0,0) \right| \leq 3/2.$$