Linear Algebra Preliminary Examinations, April 2011

1. Suppose that U, V and W are finite-dimensional linear spaces and that $S \in L(V, W)$ and $T \in L(U, V)$. Show that $\dim N_{ST} \leq \dim N_S + \dim N_T$.

2. Prove that the characteristic polynomial of a complex matrix A divides its minimal polynomial if and only if all eigenspaces of A are one-dimensional.

3. Let Y be a subspace of (not necessarily finite-dimensional) linear space X. Show that Y' (the dual of Y) is isomorphic to $X'/Y^{\perp}$.

4. For a given $n \times n$ real matrix C, consider a bilinear form F_C on the space of real $n \times n$ matrices, defined by the formula $F_C(A, B) = \mathrm{tr}(B^t \cdot C \cdot A.)$ Prove that F_C is non-degenerate if and only if C is invertible.

5. Suppose A^3 is a unitary matrix. Prove that A is diagonalizable.

6. Show that if A is a linear map on a 2-dimensional complex Euclidean space and $||A^2|| = ||A||^2$ then A is normal.